ON PROBLEM SOLVING

BY LUCIEN T. HALL Jr.

Problem solving is the process of applying previously acquired knowledge to new and unfamiliar situations, so says the National Council of Supervisors of Mathematics. Teaching students to apply previously acquired knowledge to a new and unfamiliar situation is a real problem itself.

It is necessary to <u>have knowledge</u> in order to solve a problem. One must know some basic mathematical facts, know formulas, know some non-mathematical facts, and have a good vocabulary. Of course, greater knowledge usually makes it easier to solve a problem; in fact, some situations are not problems for some folks because of their great knowledge. On the other hand, some folks with great knowledge cannot solve some problems.

Let us look at some problems and see that knowledge is needed.

(1A) What digit(s) are covered up?

```
(a)    5        (b)   53         (c)   X64
     + X            + XX               2X0
       8              81             + 18X
                                       759
```

In (a), if we have the knowledge of the addition fact 5 + 3 = 8, then the problem is rather simple. If we just have the knowledge of how to start at 5 and count to 8, keeping count on our fingers, then we can solve the problem.

In (b) a knowledge of the basic addition facts is helpful, but a knowledge that the sum 81 minus the addend 53 will give the other addend is just as valuable and we can calculate 81 - 53 with a calculator. In (c) a knowledge of several addition facts and knowing about renaming 10 tens as 1 hundred is necessary.

(1B) How many wheels are on 2 bicycles? 3, 4, 10, 10, 100, n bicycles?

Certainly, as trite as it may sound, we need the knowledge as to how many wheels are on 1 bicycle.

(1C) What is the largest number which can be written using the digits 1, 5, and 7?

A knowledge of place value is vital, and again, trite but necessary, we must know whether 1, 5, or 7 represents the largest number.

(1D) Oranges are 97¢ per dozen. What is the cost of each?

Vital knowledge is the meaning of dozen, that 97¢ can be represented as .97 and that "per" means to divide. Knowing the algorithm for dividing .97 by 12 is nice, but it can be done with a $10 handheld calculator. No calculator can substitute for knowing what "dozen" and "per" mean.

(1E) Draw a square. Using a segment, separate the square into halves in as many ways as you can.

You need to know what a segment is, and what "halves" means. And you need to know how to draw a square. Try this example. It can be done in many more than 4 ways!

To solve a problem, one must <u>be able to find information</u>. Very seldom do we have in our head all the necessary knowledge needed to solve a problem.

(2A) How old will the Empire State Building be in the year 2000? Many of us know that this problem requires subtraction, but how many know when the Empire State Building was first occupied? We must know where to find that information.

(2B) What is the weight of the water in a half-full rectangular tank 6 feet long, 2 feet wide, and 3 feet high?

Some might need to look up the formula for finding the volume of a rectangular solid. Many might approach this problem by needing to find the number of gallons per cubic foot and the weight of a gallon of water. Some might just need to find the weight of a cubic foot of water. It is most interesting that a handheld calculator will quickly do the calculations but no piece of inexpensive machinery will find information for us. We must give our students practice in finding necessary information.

How easy example 2B would be if the dimensions were given in centimetres and the mass of the water was asked for in kilograms!

Asking questions to yourself or among each other is a technique that helps in solving a problem. Asking and answering your own questions often gives you a lead.

(3A) A child takes 5 tests and has an average of 88. What score is needed on the next test to bring the average to 90?

Question: what does 88 represent? Question: where did we get 88? It (88) was the dividend when the total points were divided by 5. So we must have started with 5 X 88 or 440 points. Question: how many total points do we need to have an average of 90 on 6 tests? 6 X 90 or 540. Question: how many more points do we need on the sixth test?

Trial and error in an extremely useful technique in solving problems. We teach so much mathematics using the technique that you should get the correct answer the first time, e.g., 6 X 9 = 54, and there is no room for trial and error. When using trial and error in solving problems, it is a good idea to have a systematic approach when making the tries and wise to record the results, quite often in the form of a table. By having a systematic approach we do not miss any attempts we should have tried and by recording results we quite often see a pattern which leads to a solution.

(4A) Draw as many rectangles as you can with a perimeter of 30 (using whole numbers only).

This shortest side possible is 1, so that makes the opposite side, side C, equal 1 and the adjacent sides, B and D, each be half of 28. Let's make a table

Side A	Side B	Side C	Side D
1	14	1	14

The next rectangle could be

2	13	2	13

Begin to see a pattern? Sides A and B always have a sum of 15 so I might have a more refined table.

1 by 14

2 by 13

3 by 12

4 by 11

etc

7 by 8

8 by 7 -- this has already been used, so there seems to be 7 rectangles with perimeter of 30 (using whole numbers only).

(4B) □ + 2 = 8

For those who have knowledge and quick recall of their addition facts, this is not a problem. But for a first grader, this is a problem which can be solved by trial and error. (This not only illustrates that problem solving is a first grade objective, but it illustrates that problem solving is something more than "Word Problems.") By trial and error, even if a calculator is used, one can solve the above.

(4C) A total of 18 chickens and hogs pass through a gate. The total number of legs passing through the gate is 50. How many of each type of animal passed through the gate?

Trial and error. Make a table.

Chickens				Hogs	Total Legs
1	⟵	must add to 18	⟶	17	70 (too many legs)
2	⟵	must add to 18	⟶	16	68 (too many legs)
		etc			
7	⟵	must add to 18	⟶	11	58 (still too many leg but getting closer to 50)
		(continue table)			

My system was to start with the smallest possible number of chickens, i.e. 1, and work up.

(4D) A farmer has 100 metres of fence. What is the rectangular shaped enclosure that the farmer can use to get the greatest area of enclosed pasture?

Try. Have a system, i.e., start with one side equal to 1. Record results.

Side 1	Side 2	Side 3	Side 4	Area
1 m	49 m	1 m	49 m	49 m^2
2 m	48 m	2 m	48 m	96 m^2
3 m	47 m	3 m	47 m	141 m^2

I see that the area is increasing as I change the shape of the rectangle. I can refine my table to include just the length and width since I notice that length and width must total 50 m.

Length	Width	Area
4 m	46 m	184 m^2 (getting larger)
5 m	45 m	225 m^2 (still larger)

Finally I hit on an enclosure of 25 m X 25 m which has the greatest area, i.e., the shape is a square. (Those with more knowledge would already have known this). I wonder if that same concept would hold true if I had 320 m of fence, i.e., should I make the pasture a square 80 m by 80 m? And, for the gifted, is there a shape other than rectangular that, using the 100 m of fence, could be used to enclose a greater area than the 25 m X 25 m square?

Did you ever wonder why some smart people build square houses or round school buildings?

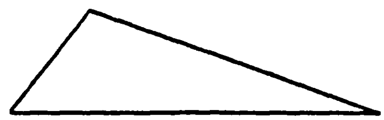
(4E) With which group of 3 letters, A - T - R or A - B - G, can you make the most words (using all 3 letters, no repetition)?

You need to try! You need a system! You need to record your trials! My system is to use a modified alphabetical order to list all of the combinations:

ART	ABG
ATR	AGB
RAT	BAG
RTA	BGA
TAR	GAB
TRA	GBA

Try this with 4 letters rather than 3.

(4F) Shade half of the triangle.



Just try! There is probably no system but you will want to record (as diagrams) your trials. A little knowledge about how the area of the triangle is found is helpful.

A most helpful technique in solving a problem is to draw a diagram. This technique must be stressed in the lower grades. Very mature mathematicians use this technique frequently in higher mathematics and there is nothing degrading about having to draw a diagram.

(5A) How many wheels are on 4 tricycles?

You need not know multiplication, i.e., 4 X 3 = 12, nor do you need to know addition, i.e., 3 + 3 + 3 + 3 = 12. Draw a diagram

and just count the wheels. This is an appropriate problem for kindergarten. The only knowledge needed is to know the number of wheels on a tricycle and how to count.

(5B) What is the sum of the degrees in the 4 angles of a 4-sided polygon? 5-sided, 6-sided, 10-sided, n-sided?

Draw a diagram of a 4-sided polygon and hope that you have the previous knowledge that the number of degrees in a triangle is 180. With a diagram of a 4-sided figure and some trial and error sketches, you can solve the problem for a 4-sided polygon. Keeping a record might reveal the pattern for finding the sum of the degrees in a 10-sided, 100-sided, or n-sided polygon.

Writing an equation or using a formula is helpful in solving some problems.

(6A) Twice a number plus 8 is 20. What is the number?

Twice	a number	plus	8	is	20
(2 X	☐)	+	8	=	20

Writing an equation quite often is a matter of translating from English to mathematical symbols. Now solve the equation, by trial and error, or by algebraic methods, depending on your knowledge.

(6B) Chickens and hogs again. A total of 18 chickens and hogs pass through a gate. The total number of legs passing through the gate is 50. How many of each type of animal passed through the gate?

Write an equation. I associate C with chickens. Chickens have 2 legs. So every time a chicken goes through the gate, 2 X C legs go through. How can I express the number of hogs? (Question to myself.) Could use H for the number of hogs and 4 X H for the number of legs per hog. Using two unknowns, C and H, I get the equation (2 X C) + (4 X H) = 50 and I must remember that C + H = 18. I could proceed to solve these equations by trial and error. Another way to approach this is as follows. Question to myself: If I have C chickens and a total of 18 animals, how many hogs? All 18 animals except the C chickens are hogs. So we have 18 - C hogs! Thus the equation: 2C + 4(18 - C) = 50. After I use this equation to find the number of chickens, I must subtract the number from 18 to find the number of hogs.

(6C) A bicycle wheel has a diameter of 26 inches. How many revolutions does the wheel make when you travel 100 feet?

You need to use a formula. I would want to know (or be able to find!) the formula for the circumference of a wheel. Some people may want to be able to calculate 3.14 X 26, calculate 12 X 100, and calculate 1200 ÷ 81.64. If my brain will hold only so much knowledge, I had rather recall $C = \pi d$, recall $\pi \approx 3.14$, and let the $10 machine (calculator) do the calculating.

<u>Starting with an easier problem of the same type</u> quite often leads us to a solution for a more difficult problem.

(7A) Chickens and hogs again. If you have a total of 18 chickens and hogs, C of which are chickens, how many hogs?

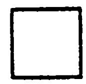
Suppose we had 3 chickens (instead of C). How many hogs? (Easier problem, same type). Not hard. 18 - 3 hogs or 15 hogs. Suppose we had 4 chickens. Not hard. 18 - 4 hogs or 14 hogs. Suppose we had 5 chickens. Not hard. 18 - 5 hogs or 13 hogs. I see the pattern. You just subtract the number of chickens. So with C chickens, not hard. 18 - C hogs or ---- 18 - C is the simplest name I can get for this answer.

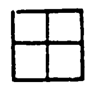
(7B) What number divided by 2/5 gives 2/5?

Stumped? Try an easier problem of the same type. What number divided by 4 gives 5. 4 X 5 = 20 and 20 is the number divided by 4 which gives 5. What divided by 6 gives 6. 6 X 6 = 36, so 36 is the number divided by 6 which gives 6. Now I see the technique. What divided by 2/5 gives 2/5. Appears to be 2/5 X 2/5 or 4/25. (Check: 4/25 ÷ 2/5 = 4/25 X 5/2 = 2/5) Got it! The key to seeing the light (getting the technique) was trying easier examples of the same type.

(7C) How many squares are on a checkerboard? (More than 64!)

A checkerboard is 8 by 8 so I immediately see (using a diagram!) 64 one-by-one squares. But how many two-by-two squares. With trial and error (the squares overlap), I see 49. This gets complicated. Let us reduce this whole thing to easier problems. Suppose the checkerboard is one-by-one. I see 1 square.

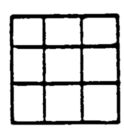
(diagram helps) 

Suppose the checkerboard is two-by-two. I see 5 squares (4 small ones and the whole thing). 

Using easier problem, trial and error, the diagram, I better keep a record.

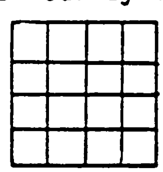
Dimensions	Squares
1 by 1	1
2 by 2	5

Suppose the checkerboard is three-by-three.



3 by 3	14

Suppose the checkerboard is four-by-four.



4 by 4	30

I see that I can finish my chart, i.e., get to 8 by 8, without drawing a diagram each time because the 3 X 3 board has 9 more squares than the preceding 5 for the 2 X 2 and 30 is 4 X 4 or 16 more than 14. So for a 5 X 5, it will be 25 more than 30, 6 X 6 will be 36 more than 55, etc. So I finish the chart and see that for an 8 X 8 checkerboard, there are 204 squares.

Dimensions	Squares	
1 X 1	1	
2 X 2	5	(2 X 2) more than 1
3 X 3	14	(3 X 3) more than 5
4 X 4	30	(4 X 4) more than 14
5 X 5	55	(5 X 5) more than 30
etc.		
7 X 7	140	(7 X 7) more than 91
8 X 8	204	(8 X 8) more than 140

I can see 204 squares when I look at an 8 X 8 checkerboard. I solved this problem by working with similiar but much easier problems of the same type.

For the want of better words, "working backwards" is a problem-solving technique. We have a tendency to use brute force to go forward; sometimes we can "back into" a solution.

(8A) How many tennis games must be played to determine a winner if 16 teams are involved?

The forward approach would be to pair off the teams, draw a diagram, let the winners of the first round play each other, etc. But let us back into this. If 16 teams play and 1 is to be the winner, then how many (question to myself!) must lose? We need 15 losers. How many games do you have to play (question!) to create 15 losers? Fifteen. So, we need to play 15 games if a winner is to be chosen from 16 starters.

Suppose we had 32 teams in the first round. How many games do we need to play to determine a winner? How easy it is to solve this now simple problem.

<u>Insight</u> is needed to solve problems. (How do you teach insight to students?)

(9A) Use the digit 8 eight times to make numbers which have a sum of 1000.

You could use considerable "trial and error" or you could use some insight and eliminate some of the trials. If the sum is 1000, then we note the 0 in the ones place of the sum. How many 8's will add up to a number with a 0 in the ones place. It takes 5 eights since 5 X 8 = 40 which gives me the 0 in the ones place.

So we record:

```
     8
     8
+    8
     8
     8
  ----
  1000  (needed sum)
```

Now more insight. So far my 5 eights give me a total of 40 and I need 1000. Insight. I better get an 8 in the hundreds place to get the sum nearer 1000.

Record:

```
     8
     8
+    8
     8
    88
  ----
  1000  (needed sum)
```

Insight. The only digit available for the tens place in the last addend is 8. So record:

```
     8
     8
+    8
     8
   888
  ----
  1000  (needed sum)
```

Insight. If I put the eighth 8 in the thousands place, the sum is too large. If I put the eighth 8 in the thousands place, then I have an empty tens place. If I put the eighth 8 in the ones place, I mess up the 0 in the sum. We better hope that the eighth 8 goes in the tens place.

Record and check.

$$\begin{array}{r} 8 \\ 8 \\ 8 \\ 88 \\ 888 \\ \hline 1000 \end{array}$$

It works.

(9B) Let us think one more time about chickens and hogs. A total of 18 chickens and hogs pass through a gate. The total number of legs passing through the gate is 50. How many of each type of animal passed through the gate?

A solution which I have heard is that you stand all 18 animals on their rear legs. This means that 36 legs are on the ground. This leaves 14 legs in the air. Only the hogs have front legs and it requires 7 hogs to have the 14 front legs. With 7 hogs, we have 11 chickens. This solution requires real insight.

Solving problems is a way to improve problem solving. Try these.

(10A) How many tires are on a school bus?

(10B) What fraction of the first 100 counting numbers end in 0?

(10C) If you buy a radio for $30, sell it for $50, buy it back for $80, and sell it again for $100, how much money have you finally gained or lost?

(10D) What is the largest whole number you can multiply by 7 and have a product that is less than 90?

(10E) Payday is Friday, November 16, 1979 and you get paid every two weeks. In what year will you receive 27 paychecks?

(10F) Instant tea cost 79¢ for 4 oz. in 1978. In 1979 instant tea costs 79¢ for 3 oz. What is the percent of increase in the cost of instant tea?

(10G) A tree has a diameter of 60 centimetres. A man cut 15 cm into the tree. A second man cut an additional 15 cm across the tree, a third man another 15 cm, and the fourth man the last 15 cm. What percent of the wood did each man cut? (It is assumed that the tree is perfectly round and that the men cut horizontally.)

(10H) A tree has a diameter of n centimetres. A man cut 0.25 of the way across the tree. What percent of the tree did the man cut?

Postscript: I am not certain where the problems used in this article originated. Some were original but I have seen Dr. Daniel S. Yates of the Mathematics and Science Center and Dr. John A. Van de Walle of Virginia Commonwealth University use some of them. Some may have come from Poyla: How to Solve It and several definitely were taken from the problem solving material produced by the Iowa Problem Solving Project at the University of Northern Iowa. The problem regarding the tree was actually sent to me for solution by a man who needed an answer.

Who knows, the techniques suggested for solving problems may be from knowledge previously acquired from other sources rather than techniques devised from facing new situations.