Powers of $\sqrt{-1}$

1) Solutions of Equations

Scientists often write mathematical expressions to describe motion. Suppose someone throws a rock straight up from a height of 6 ft at a speed of 20 $\frac{\text{ft}}{\text{s}}$. The expression $-16t^2 + 20t + 6$ gives the rock's approximate height (in feet) after t seconds. This assumes that the rock is massive enough and compact enough that air resistance affects it very little. Consider the value of $-16t^2 + 20t + 6$ at 0.5 s:

$-16(0.5)^2 + 20(0.5) + 6$	Replace t with 0.5
$-16(0.25) + 10 + 6$	Simplify
$-4 + 10 + 6$	Simplify some more
12	And some more

Thus the rock is **12 ft** above the ground 0.5 s after it was thrown.

When will the rock be at ground level? The rock will be at ground level when the expression $-16t^2 + 20t + 6$ equals 0. So we can find the value of t when the rock is at ground level by solving the equation $-16t^2 + 20t + 6 = 0$ for t. Mathematicians have a formula for solving such an equation. We get the following expression for t if we apply that formula to $-16t^2 + 20t + 6 = 0$:

$\dfrac{-20 \pm \sqrt{20^2 - 4(-16)(6)}}{2(-16)}$	The symbol "±" is read "plus or minus"
$\dfrac{-20 \pm \sqrt{400 + 384}}{-32}$	Simplify
$\dfrac{-20 \pm \sqrt{784}}{-32}$	Simplify again
$\dfrac{-20 \pm 28}{-32}$	And again
$\dfrac{8}{-32}$ or $\dfrac{-48}{-32}$	Two options
-0.25 or 1.5	Divide

The second value tells us that the rock will hit the ground **1.5 s** after it was thrown straight up. To test that conclusion, you can replace t with 1.5 (or $\frac{3}{2}$) in the expression $-16t^2 + 20t + 6$. The result should be 0.

Observe that to get the value of 1.5 s, we divided –48 by –32. So we divided a negative number by a negative number to get a positive number. People used to have a negative attitude toward negative numbers. This resulted in negative terminology being used with negative numbers. In a book published in 1557, Robert Recorde referred to a number below zero as an "*Absurde* nomber." For example, he wrote that 8 – 12 "is an *Absurde* nomber. For it betokeneth lesse then naught by 4." A copy of the paragraph that contains this terminology is reproduced below:

> **And for the thirde nomber, whiche is our speciall nomber. $\frac{648}{81}$ ——— 12. that is. 8. ———. 12. and is an *Absurde* nomber. For it betokeneth lesse then naught by. 4.**

The negative attitude people had toward numbers below zero may help to explain why such numbers received their negative name.

The fact that $\frac{-48}{-32}$ produced the useful result of 1.5 s shows that negative numbers can be quite useful. We are so used to negative numbers that we may even view them as being just as real as positive numbers. A temperature of –4°C is just as real as a temperature of 4°C. However, how should we view the expression $\frac{8}{-32}$ which gives the result of –0.25 s? Does –0.25 s have meaning or is it meaningless? Keep reading to find out!

The rock's approximate velocity (in $\frac{\text{ft}}{\text{s}}$) can be calculated using the expression $-32t + 20$ where t is the number of seconds. If t is 0.5 s, the expression $-32t + 20$ produces:

$-32(0.5) + 20$	Substitute 0.5 for t
$-16 + 20$	Simplify
4	Simplify again

Thus the rock travels upward at a speed of 4 $\frac{\text{ft}}{\text{s}}$ when it has traveled 0.5 s. If t is –0.25 s, this expression gives $-32(-0.25) + 20$ which is 8 + 20. This results in a speed of 28 $\frac{\text{ft}}{\text{s}}$. To understand this result, consider two rocks—call them *A* and *B*. Rock *A* is propelled upward at 20 $\frac{\text{ft}}{\text{s}}$ from a height of 6 ft. Rock *B* is propelled upward at 28 $\frac{\text{ft}}{\text{s}}$ from ground level. Suppose that the propulsion systems for these two rocks are next to each other with rock *B* initially at ground level and rock *A* initially 6 ft above the ground. What is required so that the rocks travel side by side? The result of –0.25 s shows that rock *B* needs to be propelled 0.25 s *before* rock *A* is propelled.

Imagine rock *B* being propelled from ground level at 28 $\frac{\text{ft}}{\text{s}}$. When it reaches 6 ft, it will have slowed down to 20 $\frac{\text{ft}}{\text{s}}$. If rock *A* is propelled at 20 $\frac{\text{ft}}{\text{s}}$ at the instant that rock *B* is 6 ft above the ground, then the two rocks will travel side by side.

Equations describe many things in this world, and mathematicians like to be able to solve equations. Consider $\boldsymbol{x^3 - 9x + 28 = 0}$. One method of solving this equation makes use of the following expression:

$$\sqrt[3]{-\frac{28}{2}+\sqrt{\left(\frac{28}{2}\right)^2+\left(\frac{-9}{3}\right)^3}}+\sqrt[3]{-\frac{28}{2}-\sqrt{\left(\frac{28}{2}\right)^2+\left(\frac{-9}{3}\right)^3}}$$

Carefully note the following steps in the simplification of the above expression:

$$\sqrt[3]{-14+\sqrt{14^2+(-3)^3}}+\sqrt[3]{-14-\sqrt{14^2+(-3)^3}}$$

$$\sqrt[3]{-14+\sqrt{196+(-27)}}+\sqrt[3]{-14-\sqrt{196+(-27)}}$$

$$\sqrt[3]{-14+\sqrt{169}}+\sqrt[3]{-14-\sqrt{169}}$$

$$\sqrt[3]{-14+13}+\sqrt[3]{-14-13}$$

$$\sqrt[3]{-1}+\sqrt[3]{-27}$$

$$-1+(-3)$$

$$\boxed{-4}$$

So **–4** is a solution to $x^3 - 9x + 28 = 0$. We can test this by evaluating $(-4)^3 - 9(-4) + 28$. This expression simplifies to $-64 + 36 + 28$ which is 0.

Now consider the equation $\boldsymbol{x^3 - 6x + 4 = 0}$. Using the same method as before, a solution of the equation can be found using the following expression:

$$\sqrt[3]{-\frac{4}{2}+\sqrt{\left(\frac{4}{2}\right)^2+\left(\frac{-6}{3}\right)^3}}+\sqrt[3]{-\frac{4}{2}-\sqrt{\left(\frac{4}{2}\right)^2+\left(\frac{-6}{3}\right)^3}}$$

Carefully note the following steps in the simplification of the above expression:

$$\sqrt[3]{-2+\sqrt{2^2+(-2)^3}}+\sqrt[3]{-2-\sqrt{2^2+(-2)^3}}$$

$$\sqrt[3]{-2+\sqrt{4+(-8)}}+\sqrt[3]{-2-\sqrt{4+(-8)}}$$

$$\sqrt[3]{-2+\sqrt{-4}}+\sqrt[3]{-2-\sqrt{-4}}$$

What should we do now? What does $\sqrt{-4}$ mean? We know that $2 \cdot 2 = 4$ and $(-2)(-2) = 4$, but what times itself equals -4? Before attempting an answer, let us consider the expression $\sqrt{12}$. This expression can be written as $\sqrt{2 \cdot 2 \cdot 3}$. Using square root arithmetic, this can be written as $2\sqrt{3}$ since $\sqrt{2 \cdot 2}=2$. Thus $\sqrt{12}=2\sqrt{3}$. Likewise, the expression $\sqrt{-4}$ can be written as $\sqrt{-1 \cdot 2 \cdot 2}$ which can be written as $2\sqrt{-1}$. Thus $\boldsymbol{\sqrt{-4}=2\sqrt{-1}}$. What does $\sqrt{-1}$ mean? It means a number that when multiplied by itself equals -1. Thus $\boxed{\boldsymbol{\sqrt{-1}\sqrt{-1}=-1}}$. ***Remember this***!

Using the fact $\sqrt{-4}=2\sqrt{-1}$, we write $\boldsymbol{\sqrt[3]{-2+\sqrt{-4}}+\sqrt[3]{-2-\sqrt{-4}}}$ as $\boldsymbol{\sqrt[3]{-2+2\sqrt{-1}}+\sqrt[3]{-2-2\sqrt{-1}}}$.

What can we do with $\sqrt[3]{-2+2\sqrt{-1}}$ and $\sqrt[3]{-2-2\sqrt{-1}}$? Let us first consider $\sqrt[3]{-2+2\sqrt{-1}}$. We will demonstrate the amazing fact that $\boldsymbol{\sqrt[3]{-2+2\sqrt{-1}}=1+\sqrt{-1}}$. To demonstrate this, we will show that $(1+\sqrt{-1})^3=-2+2\sqrt{-1}$. In showing this, we will make use of the distributive property and the fact that $\boldsymbol{\sqrt{-1}\sqrt{-1}=-1}$. Carefully observe the following steps in the simplification of $(1+\sqrt{-1})^3$:

$$(1+\sqrt{-1})(1+\sqrt{-1})(1+\sqrt{-1}) \qquad \text{Expand}$$

$$(1+\sqrt{-1}+\sqrt{-1}+\sqrt{-1}\sqrt{-1})(1+\sqrt{-1}) \quad \text{Distribute}$$

$$(1+2\sqrt{-1}+(-1))(1+\sqrt{-1}) \qquad \sqrt{-1}\sqrt{-1}=-1$$
$$(2\sqrt{-1})(1+\sqrt{-1}) \qquad \text{Simplify}$$
$$2\sqrt{-1}+2\sqrt{-1}\sqrt{-1} \qquad \text{Distribute}$$
$$2\sqrt{-1}+2(-1) \qquad \sqrt{-1}\sqrt{-1}=-1$$
$$2\sqrt{-1}-2 \qquad \text{Simplify}$$
$$-2+2\sqrt{-1} \qquad \text{Rearrange}$$

This shows that $(1+\sqrt{-1})^3=-2+2\sqrt{-1}$. Thus we can conclude that $\boldsymbol{\sqrt[3]{-2+2\sqrt{-1}}=1+\sqrt{-1}}$. Similar calculations demonstrate that $\boldsymbol{\sqrt[3]{-2-2\sqrt{-1}}=1-\sqrt{-1}}$. We can now simplify the expression $\sqrt[3]{-2+2\sqrt{-1}}+\sqrt[3]{-2-2\sqrt{-1}}$ which is one of the solutions of the equation $x^3 - 6x + 4 = 0$. Observe the amazing simplification process:

$$\boldsymbol{\sqrt[3]{-2+2\sqrt{-1}}+\sqrt[3]{-2-2\sqrt{-1}}} \qquad \text{Expression to simplify}$$
$$\boldsymbol{1+\sqrt{-1}+1-\sqrt{-1}} \qquad \text{See paragraph above}$$
$$\boxed{\mathbf{2}} \qquad \text{Simplify}$$

Thus **2** is a solution to $x^3 - 6x + 4 = 0$. We can test this by evaluating $(2)^3 - 6(2) + 4$. This expression simplifies to 8 – 12 + 4 which is 0.

<u>Example exercise</u>: Simplify $(-5+12\sqrt{-1})(2+3\sqrt{-1})$.

Solution:

$$(-5+12\sqrt{-1})(2+3\sqrt{-1}) \qquad \text{Write product}$$
$$-10-15\sqrt{-1}+24\sqrt{-1}+36\sqrt{-1}\sqrt{-1} \qquad \text{Distribute}$$
$$-10+9\sqrt{-1}+36(-1) \qquad \sqrt{-1}\sqrt{-1}=-1$$
$$-10+9\sqrt{-1}-36 \qquad \text{Simplify}$$
$$\boxed{-46+9\sqrt{-1}} \qquad \text{Simplify again}$$

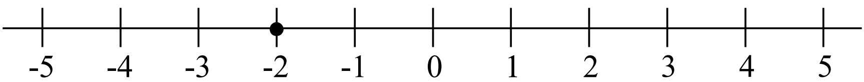
Exercise 1: Simplify $(2+3\sqrt{-1})(2+3\sqrt{-1})$.

Extra credit 1: Solve $x^3 - 24x + 72 = 0$ using the cube root method demonstrated in this lesson (study the example for $x^3 - 9x + 28 = 0$).

Extra credit 2: Solve $x^3 - 39x + 92 = 0$ using the cube root method demonstrated in this lesson (study the example for $x^3 - 6x + 4 = 0$).

2) One-dimensional and Two-dimensional Numbers

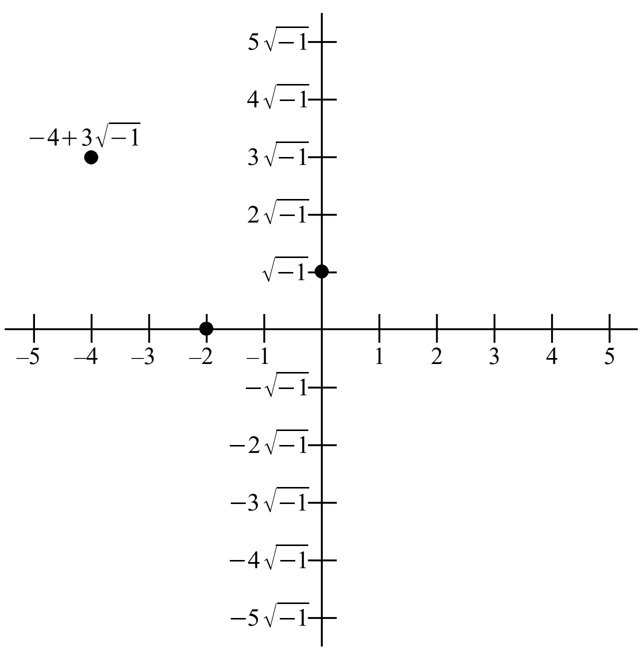
The expression $\sqrt{-1}$ was useful in helping us to discover that 2 is a solution of $x^3 - 6x + 4 = 0$. To help us get a better understanding of $\sqrt{-1}$, let us consider the number line:



The number line displayed above is a small portion of the entire number line. The entire number line continues without end in both directions. On the number line above, the number –2 has been plotted. As we study the number line, we are unable to find any number that can be multiplied by itself to make –1. Where can $\sqrt{-1}$ be found if it is not on the number line?

To find $\sqrt{-1}$, we will go to a higher dimension. The number line is one-dimensional. We will observe the *two-dimensional **number plane***.

Three points are plotted on the two-dimensional number plane shown below. These points are –2, $\sqrt{-1}$, and $-4+3\sqrt{-1}$. These

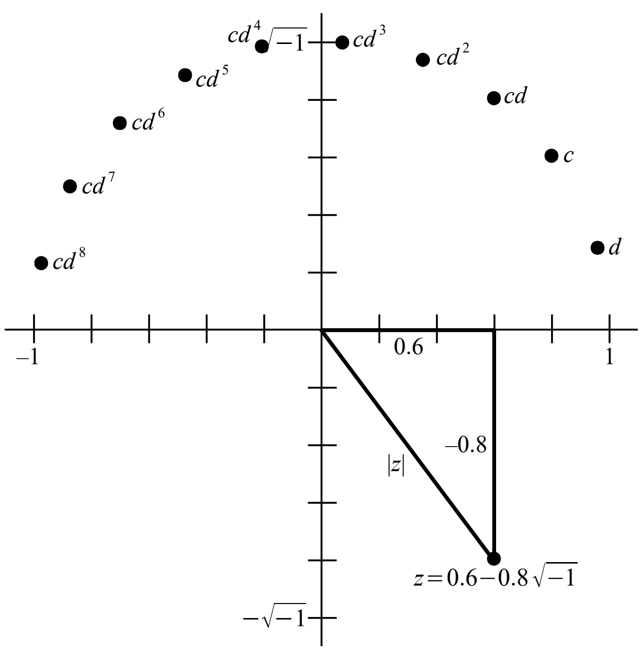


numbers are **two-dimensional numbers** (2-D #s) since they are on the two-dimensional number plane. Numbers on the number line are **one-dimensional numbers** (1-D #s).

Observe that –2 can be written as $-2+0\sqrt{-1}$. Also, $\sqrt{-1}$ can be written as $0+1\sqrt{-1}$. Every 2-D number can be written in the form $a+b\sqrt{-1}$ where a and b are numbers on the number line (1-D #s). We see that –2 can be thought of as a 1-D number or as a 2-D number. When we plot it on the number line, we think of it as a 1-D number. When we plot –2 on the number plane, we can think of it as a 2-D number. Observe that the 1-D number line is part of the 2-D number plane. Therefore every 1-D number is also a 2-D number.

You may be wondering: "Are there 3-D or 4-D numbers?" The answer to that question will come later. For now, we will focus on the astonishing properties of the 2-D numbers.

Often it is helpful to give names to numbers. Let us call c the 2-D number $0.8+0.6\sqrt{-1}$. Likewise, let $d=0.96+0.28\sqrt{-1}$. The number plane below plots these numbers as well as some other numbers. To improve our view of these numbers, the number plane



has been magnified so that each space represents a distance of 0.2 or $\frac{1}{5}$. The multiplication of 2-D numbers helps to show one reason why these numbers are so intriguing. To multiply c and d, we evaluate $(0.8+0.6\sqrt{-1})(0.96+0.28\sqrt{-1})$. When we use the distributive property we get :

$$0.8(0.96)+0.8(0.28\sqrt{-1})+0.6\sqrt{-1}(0.96)+0.6\sqrt{-1}(0.28\sqrt{-1})$$

$$0.768+0.224\sqrt{-1}+0.576\sqrt{-1}+0.168(-1) \quad \text{Simplify}$$

$$0.768-0.168+0.224\sqrt{-1}+0.576\sqrt{-1} \quad \text{Rearrange}$$

$$0.6+0.8\sqrt{-1} \quad \text{Simplify}$$

Therefore $cd=0.6+0.8\sqrt{-1}$. This point is labeled on the number plane as cd.

To find $cd{\cdot}d$ (or cd^2), evaluate $(0.6+0.8\sqrt{-1})(0.96+0.28\sqrt{-1})$. The multiplication produces the result $cd^2 = 0.352+0.936\sqrt{-1}$. We then calculate cd^3 by evaluating $cd^2{\cdot}d$. The calculations show that $cd^3 = 0.07584+0.99712\sqrt{-1}$. Evaluate $cd^3{\cdot}d$ to calculate $cd^4 = -0.2063872+0.9784704\sqrt{-1}$. We can continue with calculations to find cd^5, cd^6, cd^7, cd^8, etc. You may have noticed that the plotted points look like they lie on part of a circle. This is not an accident!

The size of a 2-D number is the distance from 0 to that number. The number $z=0.6-0.8\sqrt{-1}$ is plotted on the number plane on the previous page. A right triangle is drawn with one acute angle at 0 and the other acute angle at z. The size of z is the length of the hypotenuse of the triangle. The **size** of z is also called the **absolute value** of z and is written as $|z|$. Thus the length of the hypotenuse is $|z|$.

The length of the horizontal leg of the triangle is 0.6 and the length of the vertical leg is 0.8. To find $|z|$, use the Pythagorean theorem:

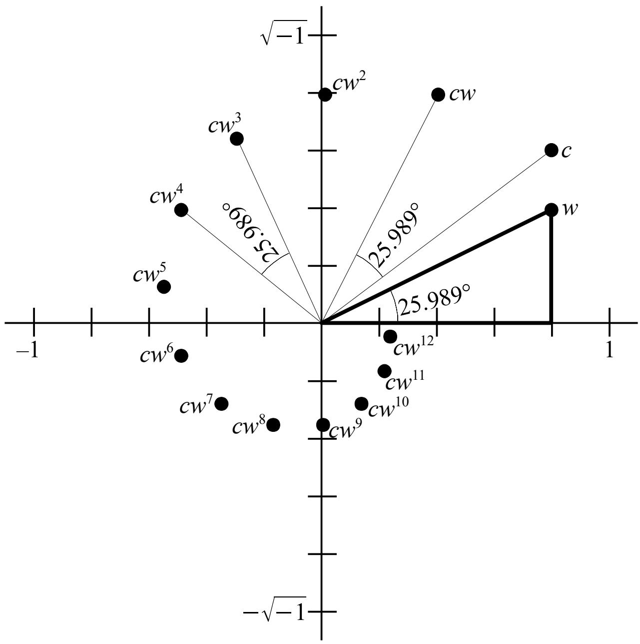
$$|z|^2 = 0.6^2 + 0.8^2 \quad \text{Pythagorean theorem}$$

$$|z|^2 = 0.36 + 0.64 \quad \text{Simplify}$$

$$|z|^2 = 1 \quad \text{Simplify again}$$

$$\boxed{|z| = 1} \quad \text{Square root both sides}$$

Now consider the number $w = 0.8 + 0.39\sqrt{-1}$. The number w is shown in the number plane below at an acute vertex of a right triangle. The number 0 is at the other acute vertex. The length of the horizontal leg of the triangle is 0.8 and the length of the vertical leg is 0.39. Since the length of the hypotenuse is $|w|$, we can use the Pythagorean theorem to write $|w|^2 = 0.8^2 + 0.39^2$. Completely simplifying this equation produces $|w|^2 = 0.7921$. If we square root both sides, we get $|w| = 0.89$.



In this number plane, $c = 0.8 + 0.6\sqrt{-1}$. Calculations show that $cw = 0.406 + 0.792\sqrt{-1}$. If we evaluate $cw \cdot w$, we discover that $cw^2 = 0.01592 + 0.79194\sqrt{-1}$. The Pythagorean theorem shows that $|c| = 1$, $|cw| = 0.89$, and $|cw^2| = 0.7921$. These numbers demonstrate an important concept about the multiplication of 2-D numbers: *The size of the product of 2-D numbers equals the product of the sizes of the numbers*. Note: $|c \cdot w| = |c| \cdot |w|$ since $0.89 = 1 \cdot 0.89$. Also, observe that $|cw \cdot w| = |cw| \cdot |w|$ since $0.7921 = 0.89 \cdot 0.89$. We also could write $|c \cdot w \cdot w| = |c| \cdot |w| \cdot |w|$ which is verified by the equation $0.7921 = 1 \cdot 0.89 \cdot 0.89$.

Recall the earlier number plane which showed that c, cd, cd^2, etc. are on part of a circle. The reason for this is that $|c| = 1$ and $|d| = 1$.

Thus any product of c and d will have a size of 1. So all the products of c and d are exactly 1 unit from 0. This means that all of these numbers lie on a circle with a radius of 1 where the center of the circle is at 0.

Since $|w| = 0.89$ which is less than 1, multiplying a 2-D number by w results in a 2-D number that is smaller than the original number. As an example, $|c \cdot w| < |c|$ since $0.89 < 1$. Also, $|cw \cdot w| < |cw|$ since $0.7921 < 0.89$. Study the diagram of the number plane on the previous page and observe that cw is closer to 0 than c. Also, cw^2 is closer to 0 than cw. As we multiply by w, we produce numbers that are closer and closer to 0.

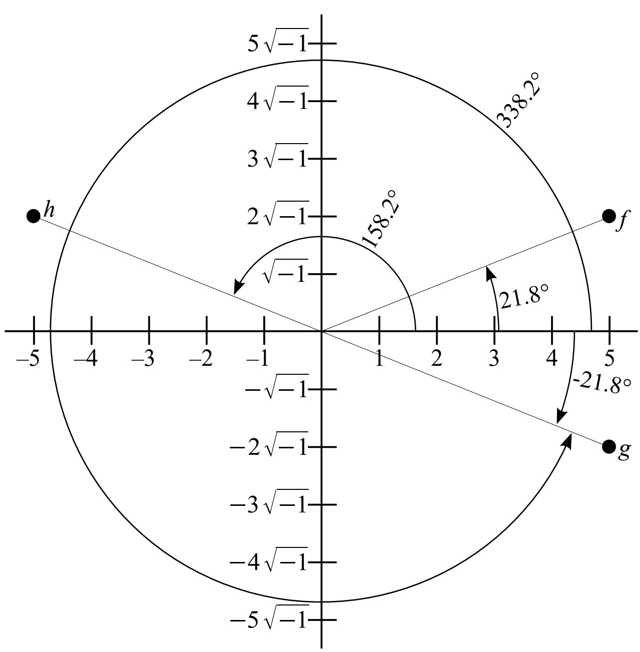
For the triangle drawn in the number plane, it can be shown that the angle at 0 is near 25.989°. If we draw line segments from 0 to c, cw, cw^3, and cw^4, it can be shown that the angle between c and cw is near 25.989°. Also, the angle between cw^3 and cw^4 is near 25.989°. Each time we multiply by w, the angle changes by approximately 25.989°. If we were to draw line segments from 0 to cw^7 and cw^8, the angle between those line segments would be near 25.989°.

As we multiply by w, we rotate counterclockwise about 0. When working with a number plane, *a **counterclockwise** rotation is a **positive** rotation*. A clockwise rotation is a negative rotation.

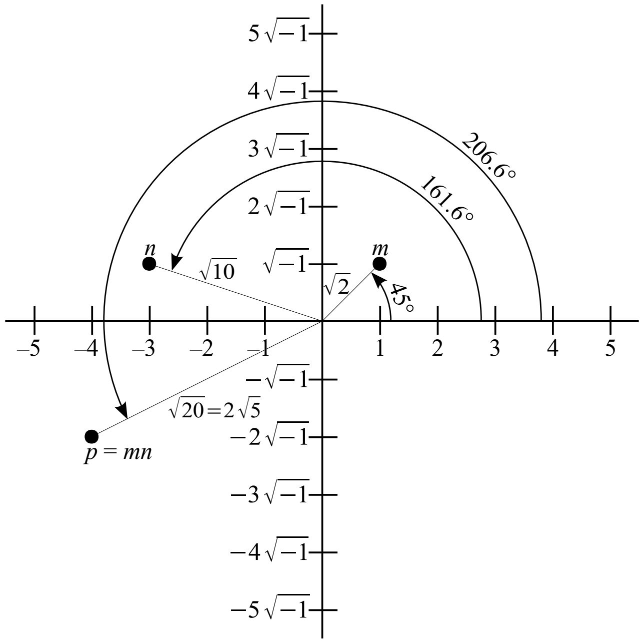
Exercise 1: To the nearest hundredth, if $s = 0.6+0.8\sqrt{-1}$, then $s^2 = -0.28+0.96\sqrt{-1}$, $s^3 \approx -0.94+0.35\sqrt{-1}$, $s^4 \approx -0.84-0.54\sqrt{-1}$, $s^5 \approx -0.08-1.00\sqrt{-1}$, and $s^6 \approx 0.75-0.66\sqrt{-1}$. We read "≈" as "is approximately equal to." On graph paper, set up a 2-D number plane where each square is 0.2 units wide (study the last two number planes printed in this lesson). The horizontal axis should vary from –1 to 1 while the vertical axis should vary from $-\sqrt{-1}$ to $\sqrt{-1}$ as shown on the previous page. On this number plane, plot and label the points s, s^2, s^3, s^4, s^5, and s^6.

3) Angles of 2-D Numbers

The number plane displayed below shows the locations of three numbers: $f = 5+2\sqrt{-1}$, $g = 5-2\sqrt{-1}$, and $h = -5+2\sqrt{-1}$. A line segment is drawn to each number. Arrows are drawn starting from the positive part of the 1-D number line. Labels show that the angle of f is around 21.8° and the angle of h is around 158.2°. The two arrows drawn to the line segment for g show that the angle of g can be thought of as –21.8° or as 338.2°. *The **angle** of a 2-D number is always measured from the positive part of the 1-D number line*. *A positive angle is a counterclockwise angle, and a negative angle is a clockwise angle*. For historical reasons, the **angle** of a 2-D number is also called the **argument** of the number. This explains why we write $\arg f \approx 21.8°$ and $\arg h \approx 158.2°$. The expression "arg f" means "the argument of f." However, we can also read "arg f" as "the angle of f" since *argument* means *angle* when we discuss 2-D numbers.



The number plane displayed below shows these three numbers: $m = 1+\sqrt{-1}$, $n = -3+\sqrt{-1}$, and $p = -4-2\sqrt{-1}$. Calculating the product $mn = (1+\sqrt{-1})(-3+\sqrt{-1})$ shows that $mn = -4-2\sqrt{-1}$. Thus $mn = p$.



The numbers in the diagram show that $|m| = \sqrt{2}$, $|n| = \sqrt{10}$, and $|mn| = \sqrt{20} = 2\sqrt{5}$. Observe that $|m|\cdot|n| = |mn|$. This verifies an important concept emphasized earlier: *The size of the product of 2-D numbers equals the product of the sizes of the numbers*.

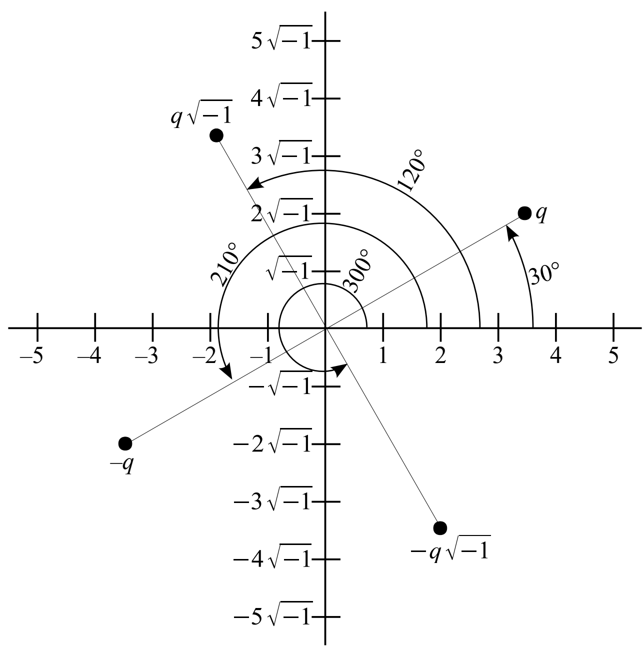
The angles of the numbers *m*, *n*, and *mn* demonstrate another important concept: *The angle of the* ***product*** *of 2-D numbers equals the* ***sum*** *of the angles of the numbers*! For the numbers plotted above, arg m = 45°, arg $n \approx 161.6°$, and arg $mn \approx 206.6°$. Observe that 206.6° = 45° + 161.6°. This demonstrates the following rule: **arg *mn* = arg *m* + arg *n*.**

Numbers on the positive 1-D number line have an angle of 0°. Numbers on the negative 1-D number line have an angle of 180°. Consider the product (–2)(–1). Since both numbers are negative 1-D numbers, they both have an angle of 180°. Thus we write arg –2 = 180° and arg –1 = 180°. According to the concept in the previous paragraph, arg (–2)(–1) = arg –2 + arg –1. Thus we can write arg (–2)(–1) = 180° + 180°. So arg (–2)(–1) = 360°. Since an angle of 360° has the same direction as an angle of 0°, we say that arg (–2)(–1) = 0°. This means that (–2)(–1) must be on the positive 1-D number line. Thus the product of (–2) and (–1) must be a positive number. We know that (–2)(–1) is the positive number 2.

A study of 2-D numbers helps us to gain a better understanding of the 1-D numbers. A study of 2-D numbers should help us to understand that multiplying by a negative number requires a 180° rotation. Since |–1| = 1 and arg –1 = 180°, multiplying by –1 will not change the size of a number. Instead, multiplying by –1 will only rotate the number by 180°.

Now consider $\sqrt{-1}$. Since $\left|\sqrt{-1}\right|=1$ and arg $\sqrt{-1}$ = 90°, multiplying by $\sqrt{-1}$ will not change the size of a number. Instead, multiplying by $\sqrt{-1}$ will only rotate a number by 90°.

Study the number $\boldsymbol{q = 2\sqrt{3}+2\sqrt{-1}}$ which is plotted on the number plane below. The angle of q is 30°. Multiplying q by $\sqrt{-1}$ produces $\boldsymbol{-2+2\sqrt{3}\sqrt{-1}}$. This multiplication rotates the number by 90° so that the angle of $\boldsymbol{q\sqrt{-1}}$ is 120°.



Multiplying $q\sqrt{-1}$ by $\sqrt{-1}$ produces $\mathbf{-2\sqrt{3}-2\sqrt{-1}}$ which has an angle of 210° since 120° + 90° = 210°. This number is plotted as ***–q*** on the number plane. Multiplying $-q$ by $\sqrt{-1}$ results in the number $\boldsymbol{-q\sqrt{-1}} = \mathbf{2-2\sqrt{3}\sqrt{-1}}$. Observe that the angle for this number is 300°.

Multiplying $2-2\sqrt{3}\sqrt{-1}$ by $\sqrt{-1}$ produces $2\sqrt{3}+2\sqrt{-1}$ which has an angle of 390° since 300° + 90° = 390°. However, 390° is the same direction as 30°. Thus the angle of $2\sqrt{3}+2\sqrt{-1}$ is also 30°. This makes good sense since $2\sqrt{3}+2\sqrt{-1}$ is the number q. Thus multiplying q by $\sqrt{-1}$ four times resulted in the number q again! Observe the pattern in the following table:

q	=	q
$q\sqrt{-1}$	=	$q\sqrt{-1}$
$q\sqrt{-1}\ \sqrt{-1}$	=	$-q$
$q\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$	=	$-q\sqrt{-1}$
$q\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$	=	q
$q\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$	=	$q\sqrt{-1}$
$q\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$	=	$-q$
$q\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$	=	$-q\sqrt{-1}$

This pattern repeats every four multiplications since arg $\sqrt{-1}$ = 90° and 4·90° = 360°.

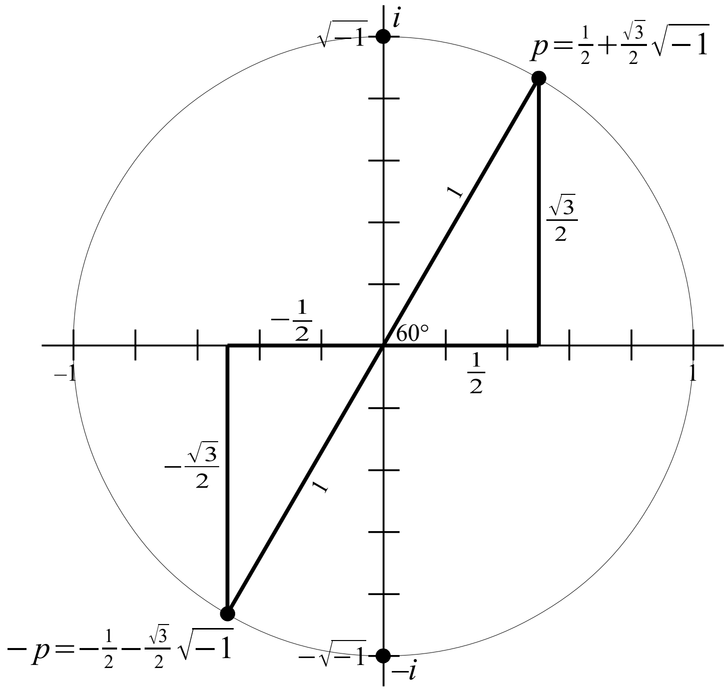
Exercise 1: Let $c = 3+4\sqrt{-1}$. Evaluate $(3+4\sqrt{-1})(\sqrt{-1})$ to find $c\sqrt{-1}$. Then find $c\sqrt{-1}\ \sqrt{-1}$. Then find $c\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$.

Exercise 2: On graph paper make a number plane such that each square is 1 unit wide. Make the horizontal axis go from –5 to 5 and the vertical axis go from $-5\sqrt{-1}$ to $5\sqrt{-1}$ (carefully study the example graphs in this lesson). Plot and label c, $c\sqrt{-1}$, $c\sqrt{-1}\ \sqrt{-1}$, and $c\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$ (use the values from the previous exercise).

Exercise 3: In the diagram on the previous page, what is the value of arg -*q*?

J) The Unit Circle

The **unit circle** is the circle centered at 0 that has a radius of 1. The number $p=\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}$ is plotted on the unit circle shown below.



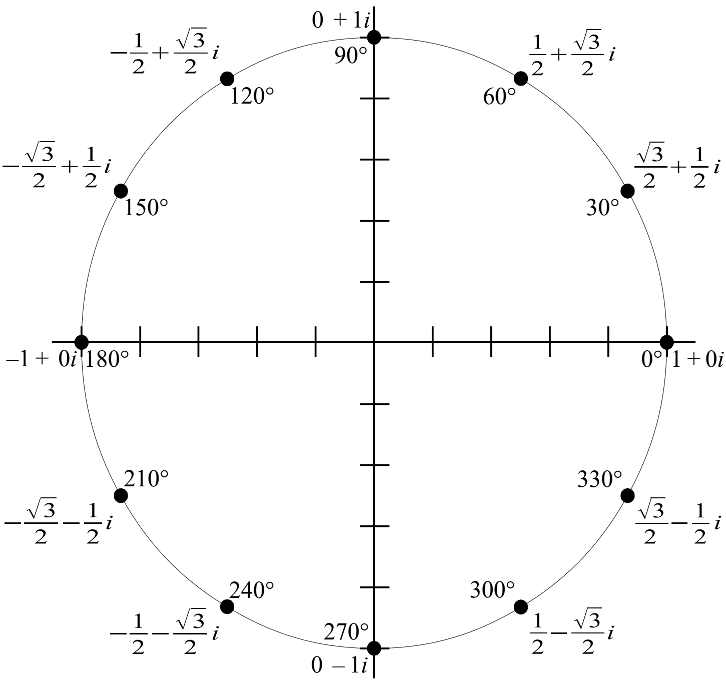
Observe that arg p = 60° and $|p|$ = 1. Using concepts from trigonometry, we say that $\cos 60°=\frac{1}{2}$ and $\sin 60°=\frac{\sqrt{3}}{2}$. Thus we can write $p=\cos 60°+\sqrt{-1}\sin 60°$.

On the diagram above, the number $\boldsymbol{i=\sqrt{-1}}$ is plotted. **It is traditional to use the symbol *i* for $\boldsymbol{\sqrt{-1}}$**. Thus $p=\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}$ can be written as $p=\frac{1}{2}+\frac{\sqrt{3}}{2}i$. Likewise, $p=\cos 60°+\sqrt{-1}\sin 60°$ can be written as $p=\cos 60°+i\sin 60°$.

The above diagram shows the number $-p=-\frac{1}{2}-\frac{\sqrt{3}}{2}\sqrt{-1}$. We can write that number as $-p=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$. From the diagram, we see that $|-p|$ = 1. Observe that arg $-p$ = 240° since 180° + 60° = 240°. In simple trigonometry, we consider the sines and cosines of angles between 0° and 90°. If we extend the ideas of sine and cosine to other angles, we discover that $\cos 240°=-\frac{1}{2}$ and $\sin 240°=-\frac{\sqrt{3}}{2}$.

Thus $-p = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ can be written as $\boxed{-p = \cos 240° + i \sin 240°}$. Compare that with $\boxed{p = \cos 60° + i \sin 60°}$. Both numbers are written as the *cosine* of an angle plus *i* times the *sine* of that angle. Each 2-D number on the unit circle can be written that way. To simplify notation, we write $\boxed{p = \text{cis } 60°}$ and $\boxed{-p = \text{cis } 240°}$. We pronounce *cis* as the word *sis*. **In *cis*, the *c* stands for *cosine*; the *i* stands for $\sqrt{-1}$; and the *s* stands for *sine*.**

Consider cis 90°. This means cos 90° + *i* sin 90°. Since cos 90° = 0 and sin 90° = 1, this shows that cis 90° = 0 + 1*i*. Thus **cis 90° = *i*.** Observe that **arg *i* = 90°** and **|*i*| = 1**.



Consider cis 270° = cos 270° + *i* sin 270°. Since cos 270° = 0 and sin 270° = –1, we see that cis 270° = 0 – 1*i*. Thus **cis 270° = – *i*.** Observe that arg –*i* = 270° and |–*i*| = 1. *Every number on the unit circle has a size of 1*. A number with a size of 1 is a **unit number**. The diagram above displays 12 special unit numbers on the unit circle along with the angle for each number. For example, the diagram shows that $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ = cis 300° which is also cis –60°.

The unit circle diagram with 12 special numbers comes in handy for certain calculations. Study the exercises below that make use of this unit circle.

Example exercise: Evaluate 8 cis 150°.

Solution:

$$8 \text{ cis } 150°$$

$$8\left(-\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right) \quad \text{See unit circle}$$

$$\boxed{-4\sqrt{3}+4i} \quad \text{Distribute}$$

Recall that if m and n are 2-D numbers, then $|m|\cdot|n| = |mn|$. Therefore, $|8 \text{ cis } 150°| = |8|\cdot|\text{cis } 150°| = 8\cdot 1 = 8$. Thus $|-4\sqrt{3}+4i| = 8$.

Also remember that arg mn = arg m + arg n. Thus we can conclude that arg (8 cis 150°) = arg 8 + arg (cis 150°) = 0° + 150° = 150°. This shows that arg $(-4\sqrt{3}+4i) = 150°$.

These calculations demonstrate that 8 cis 150° shows both the *size* and the *angle* of $-4\sqrt{3}+4i$. We say that **8 cis 150°** is the **polar** form of the 2-D number while $\boldsymbol{-4\sqrt{3}+4i}$ is the **rectangular** form of the 2-D number. **When writing in rectangular form, always put the *i* term last**. Do not forget the basic concepts shown by the equations |8 cis 150°| = 8 and arg (8 cis 150°) = 150°.

Example exercise: Write the rectangular form of 6 cis 390°.

To solve, make use of the fact that a circle has 360°. Thus cis 30° is at the same location as cis 390° since 30° = 390° – 360°.

Solution:

$$6 \text{ cis } 390°$$

$$6 \text{ cis } 30° \quad \text{cis } 390° = \text{cis } 30°$$

$$6\left(\tfrac{\sqrt{3}}{2}+\tfrac{1}{2}i\right) \quad \text{See unit circle}$$

$$\boxed{3\sqrt{3}+3i} \quad \text{Distribute}$$

Exercise 1: Write the rectangular form of 10 cis –30°.

Exercise 2: Study the unit circle. What is arg –*i*?

Exercise 3: Study the unit circle. What is arg $\left(\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\right)$?

5) Multiplying 2-D Numbers

<u>*Example exercise*</u>: Write both the polar form and the rectangular form of the product (3 cis 120°)(2 cis 330°).

Solution: First consider the size of the product. Make use of the fact that $|mn| = |m|\cdot|n|$ to do these calculations:

$$|(3 \text{ cis } 120°)(2 \text{ cis } 330°)| = |3 \text{ cis } 120°|\cdot|2 \text{ cis } 330°| = 3\cdot 2 = \underline{6}.$$

Now consider the angle of the product. Make use of the fact that arg mn = arg m + arg n. Thus arg [(3 cis 120°)(2 cis 330°)] = arg (3 cis 120°) + arg (2 cis 330°) = 120° + 330° = <u>450°</u>.

The above calculations show that 6 cis 450° is a polar form of the product. However, *an angle of 360° or higher is not acceptable in an answer.* Thus subtract 360° from 450° to get an angle of 90°. Thus 6 cis 450° = $\boxed{6 \text{ cis } 90°}$ = 6 (0 + 1i) = $\boxed{6i}$ (see unit circle).

Since $i = \sqrt{-1}$, it follows that $\boldsymbol{i^2} = ii = \sqrt{-1}\,\sqrt{-1} = \mathbf{-1}$. Therefore we write $i^3 = i^2\cdot i = -1\cdot i = -i$. Then we conclude that $i^4 = i^2\cdot i^2 = -1\cdot(-1) = 1$. Therefore $i^5 = i^4\cdot i = 1\cdot i = i$ and $i^6 = i^4\cdot i^2 = 1\cdot(-1) = -1$. The results are in the table to the right. Study the pattern in the table. This pattern repeats every 4th power of i since multiplying by i rotates a number by 90° without changing the size of the number. Remember that **arg *i* = 90°** and **|*i*| = 1**.

i^1	$= i$
i^2	$= -1$
i^3	$= -i$
i^4	$= 1$
i^5	$= i$
i^6	$= -1$
i^7	$= -i$
i^8	$= 1$

<u>*Example exercise*</u>: Evaluate $(-2 + 3i)(4 - 5i)$.

Solution:

$(-2 + 3i)(4 - 5i)$	
$(-2)(4) + (-2)(-5i) + 3i(4) + 3i(-5i)$	Distribute
$-8 + 10i + 12i - 15i^2$	Simplify
$-8 + 22i - 15(-1)$	$\boldsymbol{i^2 = -1}$
$-8 + 22i + 15$	Simplify
$\boxed{7 + 22i}$	Simplify

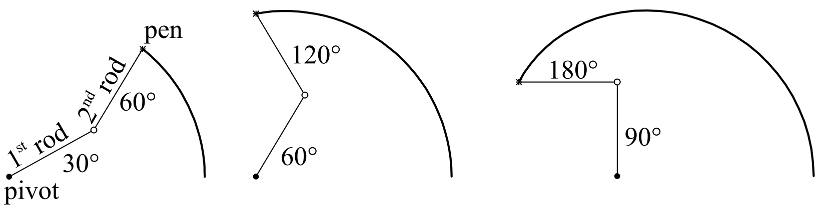
Exercise 1: Write both the polar and rectangular forms of the product (4 cis 150°)(3 cis 240°).

Exercise 2: Evaluate $(1 - 2i)(-3 + 4i)$.

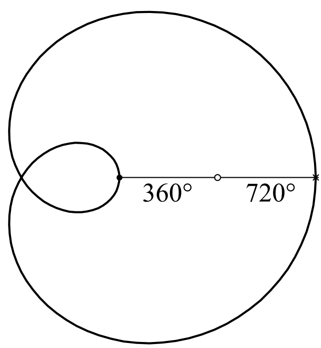
6)

Rotating Rods

Consider two equally long rotating rods in which the 2nd rod rotates *twice as fast* as the 1st rod. Also, the 2nd rod is attached to the end of the 1st rod and has a pen attached to its free end. In the diagrams below, the 1st rod attaches to the plane at the circle with the black interior, and the 2nd rod attaches to the 1st rod at the circle that has a white interior. The curve shows the path traced by the pen.



Both rods start at 0°. The leftmost diagram above shows the path traced by the pen as the 1st rod rotates through 30° and the 2nd rod rotates through 60°. The rightmost diagram above shows the path traced by the pen as the 1st rod rotates through 90° and the 2nd rod rotates 180°. The diagram to the right of this paragraph shows the curve that results when the 1st rod rotates 360° and the 2nd rod rotates twice for a total of 720°. As the rods continue to rotate, they will retrace the curve that has already been drawn.



We will use the lowercase Greek letter θ (theta) for measures of angles. In the leftmost diagram above, $\theta = 30°$ and $2\theta = 60°$. The curve sketched by these rotating rods can be described using the equation $\mathbf{z = \text{cis}\ \theta + \text{cis}\ 2\theta}$.

<u>*Example exercise*</u>: Find z when $\theta = 30°$.

Solution:

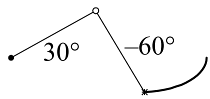
$\text{cis}\ 30° + \text{cis}\ 2(30°)$	Substitute 30° for θ
$\text{cis}\ 30° + \text{cis}\ 60°$	Simplify
$\frac{\sqrt{3}}{2} + \frac{1}{2}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i$	See unit circle
$\boxed{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)i}$	Rearrange

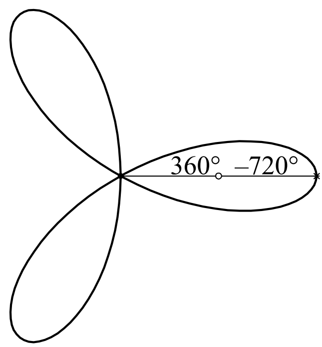
Example exercise: Find z when $\theta = 60°$.

Solution:

$\text{cis } 60° + \text{cis } 2(60°)$	Substitute 60° for θ
$\text{cis } 60° + \text{cis } 120°$	Simplify
$\frac{1}{2}+\frac{\sqrt{3}}{2}i+-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	See unit circle
$\boxed{\sqrt{3}i}$	Simplify

Consider the equation $\boldsymbol{z = \text{cis } \theta + \text{cis } {-2\theta}}$. This describes a setup where the 2nd rod rotates opposite the direction the first rod rotates. The one diagram shows the curve drawn when the 1st rod has rotated 30° and the 2nd rod has rotated –60°. The other diagram shows the curve after the 1st rod rotates 360° and the 2nd rod rotates –720°.





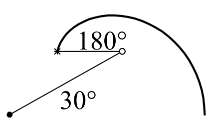
Example exercise: Find z when $\theta = 60°$.

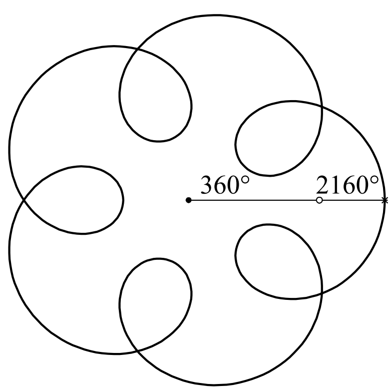
Solution:

$\text{cis } 60° + \text{cis } {-2(60°)}$	Substitute 60° for θ
$\text{cis } 60° + \text{cis } {-120°}$	Simplify
$\text{cis } 60° + \text{cis } 240°$	$-120° + 360° = 240°$
$\frac{1}{2}+\frac{\sqrt{3}}{2}i+-\frac{1}{2}-\frac{\sqrt{3}}{2}i$	See unit circle
$\boxed{0}$	Simplify

This shows that when $\theta = 60°$, the pen draws the point $0 + 0i$.

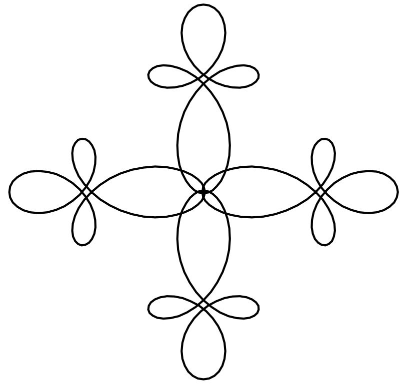
The equation $\boldsymbol{z = 2 \text{ cis } \theta + \text{cis } 6\theta}$ describes a setup where the 1st rod is 2 units long and the 2nd rod is 1 unit long. The 2nd rod rotates 6 times as fast as the 1st rod. The one diagram shows the curve drawn when the 1st rod rotates through 30° and the 2nd rod rotates 180°. The other diagram shows the curve after the 1st rod rotates 360° and the 2nd rod rotates 2160°.



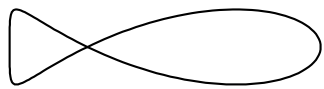


<u>*Example exercise*</u>: Find z when $\theta = 150°$ (for last design on previous page).

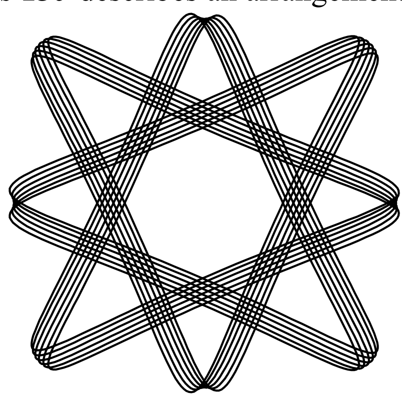
Solution: (for **z = 2 cis θ + cis 6θ**)



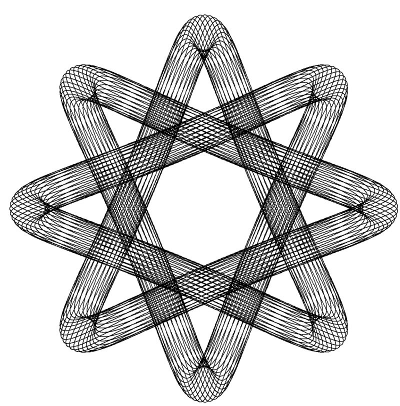
$2\text{ cis }150° + \text{cis }6(150°)$	Substitute
$2\text{ cis }150° + \text{cis }900°$	Simplify
$2\text{ cis }150° + \text{cis }180°$	900° – 720°
$2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)+-1+0i$	Unit circle
$-\sqrt{3}+1i-1$	Simplify
$\boxed{\left(-1-\sqrt{3}\right)+i}$	Rearrange

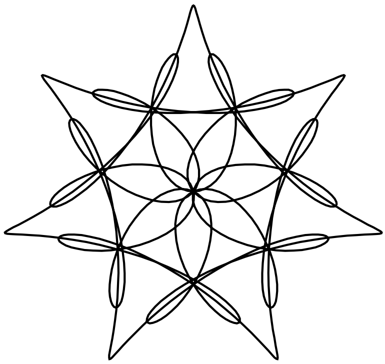


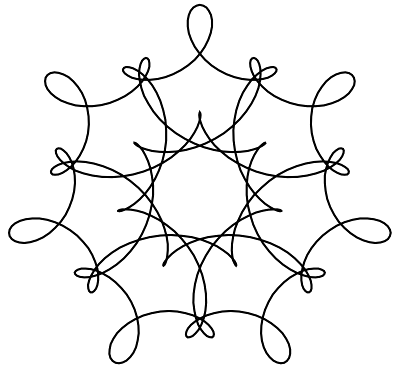
Now consider three connected rods. The 1st rod attaches to the plane at 0. The 2nd rod attaches to the free end of the 1st rod, and the 3rd rod attaches to the free end of the 2nd rod. The pen drawing the curve is on the 3rd rod. The equation **z = 3 cis θ + 2 cis -3θ + cis 13θ** describes an arrangement where the 3rd rod rotates 13 times as fast as the 1st rod. The top diagram shows the curve for this equation as θ varies from 0° to 360°.



The equation for the fish design is **z = 2 cis θ + 2 cis -θ + cis 2θ**. The equation for the design to the right: **z = 7 cis 3θ + 3 cis -5θ – cis -1.04θ**. The minus sign before the 3rd *cis* shows that the 3rd rod points to the left when θ = 0°. To produce this design, θ was varied from -1077° to 1077°. Since -1.04 is not an integer, the -1.04 keeps the design from repeating every 360°. The complete design repeats every 9000°. A slightly zoomed out view drawn with a thinner curve is at the bottom. It was drawn by varying θ from -4500° to 4500°.







The equations for the two designs above:

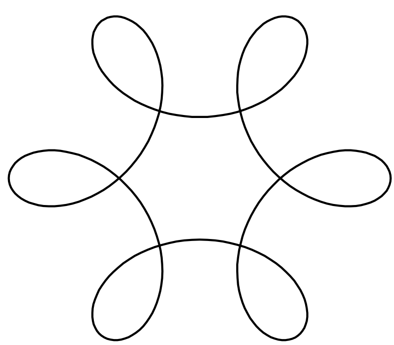
$$z = 2i \text{ cis } 4\theta + i \text{ cis } 11\theta + i \text{ cis } {-17\theta} \quad \text{(left design)}$$

$$z = 4i \text{ cis } 4\theta + 2i \text{ cis } {-3\theta} + i \text{ cis } {-24\theta} \quad \text{(right design)}$$

For each equation above, each *cis* has an i in front of it. Multiplying each *cis* by i rotates each arm by 90°. Thus when $\theta = 0°$, each arm points *up* instead of to the *right*. This results in rotating the entire design by 90°.

The rotating rods concept is behind two designs on the title page. The equation $\boldsymbol{z = 12 \text{ cis } 1.004\theta + 5 \text{ cis } {-7\theta} + 2 \text{ cis } 25\theta}$ where θ varies from -719° to 719° describes the top such design. The equation $\boldsymbol{z = 13 \text{ cis } \theta + 6 \text{ cis } {-7.01\theta} + 3 \text{ cis } {-23\theta}}$ where θ varies from –720° to 720° describes the bottom such design.

Exercise 1: The equation for this design is $z = 2 \text{ cis } \theta + \text{cis } {-5\theta}$. Find z when $\theta = 60°$.

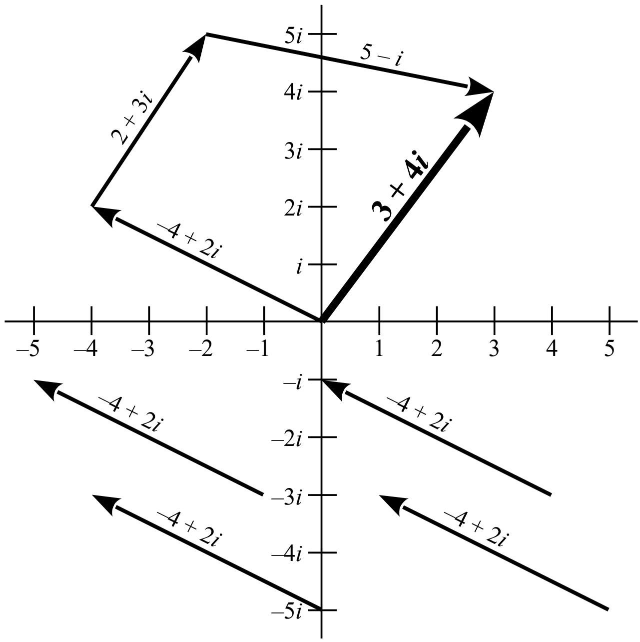


Exercise 2: Write both the polar and rectangular forms of the product (cis 60°)(2 cis 150°).

Exercise 3: Evaluate i^3 (for assistance, consider $i^2 \cdot i$).

7) Arrows

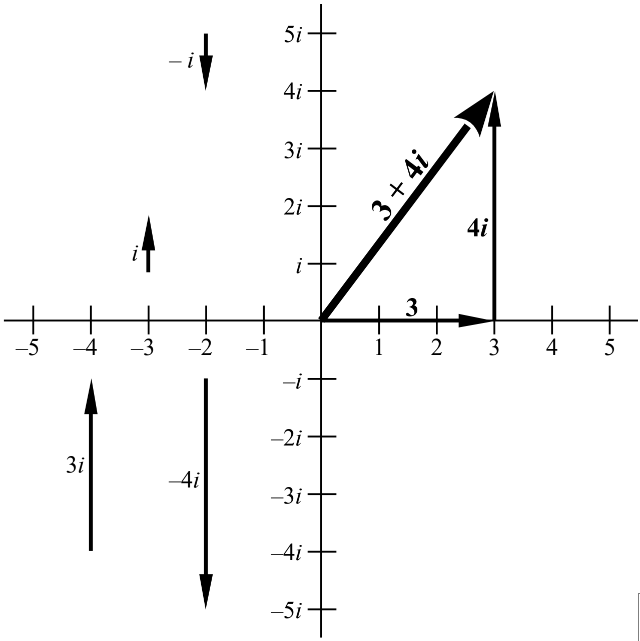
We have viewed 2-D numbers as being points in the 2-D number plane. We can also view 2-D numbers as arrows. The number plane below shows the number –4 + 2*i* written beside some arrows. One arrow begins at the number 0 and ends at –4 + 2*i*. Another arrow begins at 5 – 5*i* and ends at 1 – 3*i*. We say that its **tail** is at 5 – 5*i* and its **head** is at 1 – 3*i*. Each arrow labeled as –4 + 2*i* points in the same direction and has the same size. Thus each one represent the same number: –4 + 2*i*.



Arrows that represent 2-D numbers are called **vectors**. When we slide a vector without changing its direction or size, we do not change the number the vector represents. All of the –4 + 2*i* vectors are parallel vectors.

Consider the vector that starts at 0 and ends at –4 + 2*i*. To the head of that vector we attach the tail of the vector 2 + 3*i*. To the head of the vector 2 + 3*i* we attach the tail of the vector 5 – *i*. When **we draw a vector from 0 to the *head* of 5 – *i***, we get the vector 3 + 4*i*. This is the value of (–4 + 2*i*) + (2 + 3*i*) + (5 – *i*). You have just observed the **head-to-tail** method of adding –4 + 2*i* and 2 + 3*i* and 5 – *i*.

The number plane below shows vectors 3 and $4i$. When we use the head-to-tail method to add those vectors, we get $3 + 4i$. Since $4i$ is perpendicular to the 1-D number line, we will call $4i$ the **perpendicular portion** of the number $3 + 4i$. We will call 3 the **1-D portion** of the number $3 + 4i$.



If the vector for a number is perpendicular to the number line, we will call that number a **perpendicular number**. The number plane above shows vectors for the following perpendicular numbers: i, $-i$, $3i$, $-4i$, and $4i$.

Recall the table to the right. Consider $(-i)^2$. This equals $(-i)(-i) = (i^3)(i^3) = i^6 = -1$. So both i^2 and $(-i)^2$ equal -1. In the number plane, i and $-i$ are the only perpendicular numbers with a size of 1. We will call them **unit perpendicular numbers**. Observe that *the square of a unit perpendicular number is –1*.

i^1	$= i$
i^2	$= -1$
i^3	$= -i$
i^4	$= 1$
i^5	$= i$
i^6	$= -1$
i^7	$= -i$
i^8	$= 1$

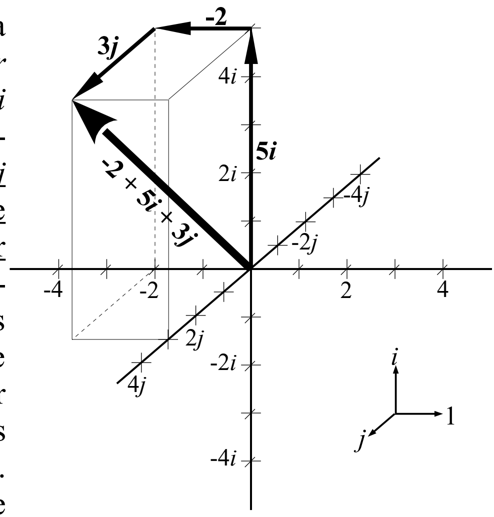
Exercise 1: On graph paper, set up a number plane where the horizontal axis varies from –5 to 5 and the vertical axis varies from $-5i$ to $5i$. Use the head-to-tail method to add –4 and $5i$. On the same number plane, use the head-to-tail method to add the three vectors $3 - 5i$, $2 + 7i$, and $-1 + i$. *Label every vector.*

8)

4-D Numbers

Since 2-D numbers are so useful, might there be 3-D numbers that can help us do 3-D work? There are numbers beyond 2-D numbers that help us with 3-D calculations. However, these are 4-D numbers! There are no 3-D numbers, but there are a couple of types of 4-D numbers. We will focus on only one type.

Recall that $\boldsymbol{i^2 = -1}$ and i is a *unit perpendicular number* since $|i| = 1$ and the vector i is perpendicular to the number line. Consider a vector j that is perpendicular to the number line and to the vector i. The diagram to the right attempts to show 3-D concepts on a 2-D page. It shows the vectors 1, i, and j with their tails at $3 - 3i$. The vector j is perpendicular to both 1 and i. Vector j points out of the page. Its head is at $3 - 3i + j$.

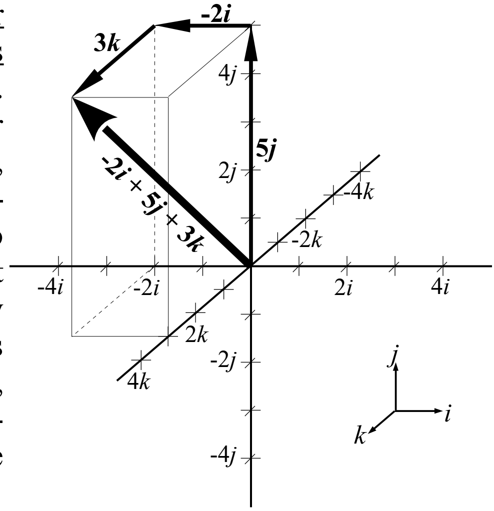


To increase understanding, look at the vector $-2 + 5i + 3j$. This vector also points out of the page, but at a different direction than vector j. The vectors $5i$, -2, and $3j$ are all perpendicular to each other. The diagram shows the head-to-tail method being used to add them to make the vector $-2 + 5i + 3j$.

Observe that $|j| = 1$. Thus j is a unit perpendicular number since it is perpendicular to the number line. Vector j is not in the 2-D number plane since it is perpendicular to the 2-D number plane. Thus it does not equal either i or $-i$ which are the unit perpendicular numbers in the 2-D number plane. Recall that *the square of a unit perpendicular number is –1*. Thus $\boldsymbol{j^2 = -1}$.

When we have a system of numbers, we normally want to be able to multiply the numbers. Thus we want to be able to calculate the value of ij. However, the numbers we have so far do not allow us to calculate ij. To find that value, we need the number k.

<u>The vector *k* is perpendicular to the number line and is perpendicular to both *i* and *j*.</u> Thus to display the number line along with vectors *i*, *j*, and *k* would require four dimensions! The diagram to the right does not attempt such a thing. Instead, it only shows the three dimensions that contain the vectors *i*, *j*, and *k*. It leaves out the dimension that contains the number line.



In this diagram, vectors *i* and *j* are in the same plane as the page. Vector *k* points out of the page. Those three vectors are drawn with their tails at $3i - 3j$. The head of vector *k* is at $3i - 3j + k$. The diagram shows the head-to-tail method being used to add $5j$, $-2i$, and $3k$ to make $-2i + 5j + 3k$.

We now have 4-D numbers. We can write $4 - 2i + 5j + 3k$ as an example of a 4-D number. We cannot draw in four dimensions, but we can still write 4-D numbers. The numbers 4, $-2i$, $5j$, and $3k$ are all perpendicular to each other. The number 4 is on the number line while $-2i$, $5j$, and $3k$ are all perpendicular to the number line. Thus $-2i$, $5j$, $3k$, and $-2i + 5j + 3k$ are **perpendicular numbers**. The **perpendicular portion** of $4 - 2i + 5j + 3k$ is the perpendicular number $-2i + 5j + 3k$. The **1-D portion** of $4 - 2i + 5j + 3k$ is 4.

Observe that $|k| = 1$ and *k* is perpendicular to the number line. Thus *k* is a unit perpendicular number. Therefore $\boldsymbol{k^2 = -1}$. We are finally able to evaluate *ij*. The answer is *k*. For these 4-D numbers to act as we think numbers should act, we will need to accept an idea that is different from an idea we use in elementary arithmetic. To see this new idea, study the multiplication facts shown below:

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j$$

Observe that $ji = -ij$ and $kj = -jk$ and $ik = -ki$. Changing the order of multiplication changes the answer! When we multiply the numbers i, j, and k, the order of multiplication is important. This different concept may seem strange, but it is useful in helping us to understand how God's creation operates. The tables to the right should help you to multiply 4-D numbers.

Study these tables

$i^2 = -1$	$j^2 = -1$	$k^2 = -1$

$ij = k$	$ji = -k$
$jk = i$	$kj = -i$
$ki = j$	$ik = -j$

<u>*Example exercise*</u>: Simplify $(4i - 3k)(1 + 2k)$.

Solution:

$(4i - 3k)(1 + 2k)$	
$4i(1) + 4i(2k) - 3k(1) - 3k(2k)$	Distribute
$4i + 8(-j) - 3k - 6(-1)$	$\boldsymbol{ik = -j}$ and $k^2 = -1$
$4i - 8j - 3k + 6$	Simplify
$\boxed{6 + 4i - 8j - 3k}$	Rearrange

<u>*Example exercise*</u>: Simplify $(1 + 2k)(4i - 3k)$.

Solution:

$(1 + 2k)(4i - 3k)$	
$1(4i) + 1(-3k) + 2k(4i) + 2k(-3k)$	Distribute
$4i - 3k + 8(j) - 6(-1)$	$\boldsymbol{ki = j}$ and $k^2 = -1$
$4i - 3k + 8j + 6$	Simplify
$\boxed{6 + 4i + 8j - 3k}$	Rearrange

Observe that $(4i - 3k)(1 + 2k) \neq (1 + 2k)(4i - 3k)$.

<u>*Example exercise*</u>: Find $|2 - i - 4j + 3k|$.

Solution: To find the absolute value of a 4-D number, extend the Pythagorean theorem to four dimensions. Let $q = 2 - i - 4j + 3k$. The four components of q are 2, $-i$, $-4j$, and $3k$. Find the absolute value of each component of q: $|2| = 2$, $|-i|=1$, $|-4j| = 4$, and $|3k| = 3$. Then $|q|^2 = 2^2 + 1^2 + 4^2 + 3^2$. Thus $|q|^2 = 4 + 1 + 16 + 9 = 30$. Therefore $|q| = \boxed{\sqrt{30}}$.

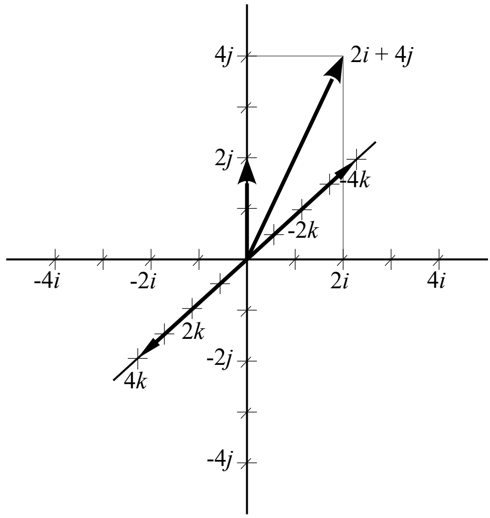
Exercise 1: Simplify $(2j - 5k)(3 + j)$.

Exercise 2: Find $|1 + 2i - j - 3k|$.

9)

3-D Vectors

Consider $-2i + 5j + 3k$. That number can be written in the form $0 - 2i + 5j + 3k$. Thus the value of the 1-D portion of that number is 0. We will call $-2i + 5j + 3k$ a **3-D vector**. *A 3-D vector is a 4-D number in which the 1-D portion equals 0*. The diagram to the right shows the 3-D vectors $2j$, $2i + 4j$, $4k$, and $-4k$. For each of these 4-D numbers, the value of the 1-D portion is 0.



3-D vectors are powerful tools used in the study of God's creation. Often, 3-D vectors are printed using bold type. Thus $2i + 4j$ may be printed as $2\mathbf{i} + 4\mathbf{j}$. When written by hand, $2i + 4j$ may be written as $2\vec{\imath} + 4\vec{\jmath}$. The half-arrows show that i and j are vectors. These notations will not be used here.

Observe the results of multiplying $2i + 4j$ and $2j$ in different orders:

$\mathbf{(2i + 4j)(2j)}$	$\mathbf{(2j)(2i + 4j)}$	Multiply 2 ways
$4ij + 8j^2$	$4ji + 8j^2$	Distribute
$4k + 8(-1)$	$4(-k) + 8(-1)$	$ij = k$ and $ji = -k$ and $j^2 = -1$
$\mathbf{-8 + 4k}$	$\mathbf{-8 - 4k}$	Simplify and rearrange

The 4-D products are not 3-D vectors since the 1-D portion equals -8 instead of 0 for each product. Because of such results, we do not think of 3-D vectors as being 3-D numbers. However, the 4-D products of 3-D vectors are still quite useful. If we extract the *perpendicular portion* of a product of two 3-D vectors, then that portion by itself is a 3-D vector. We call that resulting vector the **cross product** of the two vectors. To indicate the cross product, we use the symbol ×. So $\mathbf{(2i + 4j)\times(2j) = 4k}$ and $\mathbf{(2j)\times(2i + 4j) = -4k}$. *A cross product of two vectors is perpendicular to both of the original vectors unless the cross product is 0*. In the diagram above, both $4k$ and $-4k$ are perpendicular to $2i + 4j$ and $2j$.

Recall that $(2i + 4j)(2j) = -8 + 4k$ and $(2j)(2i + 4j) = -8 - 4k$. Each time, the 1-D portion of the product is –8. The negative of the 1-D portion of the product of two 3-D vectors is quite useful in science. We call the *negative* of the *1-D portion* of the product of two vectors the **dot product** of the vectors. We will use a large dot to indicate the dot product. Thus we write $\mathbf{(2i + 4j)\bullet(2j) = 8}$. We can also write $\mathbf{(2j)\bullet(2i + 4j) = 8}$. *If the dot product is 0, then the vectors are perpendicular to each other (if neither original vector is 0).*

Example exercise: Find $(2i + j - k)\times(3k)$ and $(2i + j - k)\bullet(3k)$.

Solution:

$(2i + j - k)(3k)$	Find 4-D product
$6ik + 3jk - 3k^2$	Distribute
$6(-j)+3(i) - 3(-1)$	$ik = -j$ and $jk = i$ and $k^2 = -1$
$3 + 3i - 6j$	Simplify and rearrange

Thus $\boxed{(2i + j - k)\times(3k) = 3i - 6j}$ and $\boxed{(2i + j - k)\bullet(3k) = -3}$.

Remember that the cross product is the perpendicular portion of the 4-D product, and the dot product is the negative of the 1-D portion.

Example exercise: Find $i\times j$, $i\bullet j$, $i\times i$, and $i\bullet i$.

Solution:

ij	ii	Find 4-D products
k	-1	$ij = k$ and $i^2 = -1$

Thus $\boxed{i\times j = k}$, $\boxed{i\bullet j = 0}$, $\boxed{i\times i = 0}$, and $\boxed{i\bullet i = 1}$.

For the product ij, the answer has only a perpendicular portion. Thus the value of the 1-D portion is 0 and $i\bullet j = 0$. Since i and j are perpendicular to each other, the dot product had to be 0. For the product ii, the answer has only a 1-D portion. Thus the value of the perpendicular portion is 0 and $i\times i = 0$. The result $i\times i = 0$ demonstrates the rule that *the cross product of parallel vectors is 0*.

If you need to find only a dot product or only a cross product, then you can do only the calculations that will affect the answer. Experience in finding dot products and cross products will help you to learn which calculations affect which products.

Exercise 1: Find $(2j)\times(i - 3j + 2k)$ and $(2j)\bullet(i - 3j + 2k)$.

10) Unit Numbers

Recall that $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ is on the unit circle and is a unit number since $|\frac{1}{2}-\frac{\sqrt{3}}{2}i|=1$. Sometimes we want a unit number or vector that points in the same direction as another number. To find such a number, *divide the original number by its absolute value*.

Example exercise: Find a unit vector that points in the same direction as $3i - 2j + 6k$.

Solution: If $v = 3i - 2j + 6k$, then $|v|^2 = 3^2 + (-2)^2 + 6^2 = 9 + 4 + 36 = 49$. Since $|v|^2 = 49$, $|v| = 7$. Therefore $\frac{v}{|v|} = \boxed{\frac{3}{7}i-\frac{2}{7}j+\frac{6}{7}k}$. Calculations show that $|\frac{3}{7}i-\frac{2}{7}j+\frac{6}{7}k| = 1$.

The vector $3i - 2j + 6k$ is **parallel** to the unit vector $\frac{3}{7}i-\frac{2}{7}j+\frac{6}{7}k$.

Example exercise: Find a unit vector that points in the same direction as $4i + 2k$.

Solution: If $v = 4i + 2k$, then $|v|^2 = 4^2 + 2^2 = 16 + 4 = 20$. Therefore $|v|=\sqrt{20}=\sqrt{4\cdot 5}=2\sqrt{5}$. Thus $\frac{v}{|v|}=\frac{4}{2\sqrt{5}}i+\frac{2}{2\sqrt{5}}k=\frac{2}{\sqrt{5}}i+\frac{1}{\sqrt{5}}k$. To rationalize the denominators, we multiply each term by $\frac{\sqrt{5}}{\sqrt{5}}$. Thus the unit vector is $\boxed{\frac{2\sqrt{5}}{5}i+\frac{\sqrt{5}}{5}k}$.

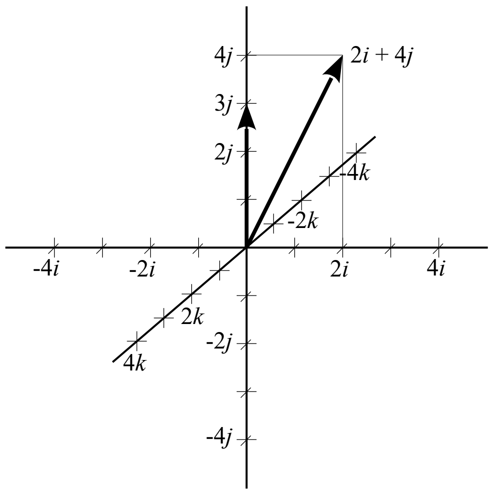
Exercise 1: Find a unit vector that points in the same direction as $i + 3j - 2k$.

Exercise 2: Find $(i + 3j - 2k)\times(2i)$ and $(i + 3j - 2k)\bullet(2i)$.

33) Vector Angles

Sometimes we want to know the angle between two 3-D vectors. The dot product helps us to find that angle. Consider the vectors $u = 3i - j + 2k$ and $v = 4i + 5j - 7k$. Careful calculations show that $u \bullet v = 3(4) + (-1)(5) + 2(-7) = 12 - 5 - 14 = -7$. In words, *multiply coefficients of like terms and add the products*.

Consider the vectors shown in the diagram to the right. Let $p = 3j$ and $q = 2i + 4j$. The only like terms are $3j$ and $4j$. Thus $p \bullet q = 3(4) = \underline{12}$.



In the formula below, θ is the angle between p and q:

$$\cos\theta = \frac{p \bullet q}{|p||q|}$$

Obviously, $|p| = |3j| = \underline{3}$.
To find $|q|$, first find $|q|^2$:
$|q|^2 = 2^2 + 4^2 = 4 + 16 = 20$.
So $|q| = \sqrt{20} = \sqrt{4 \cdot 5} = \underline{2\sqrt{5}}$.

According to the formula above, $\cos\theta = \frac{12}{3(2\sqrt{5})} = \frac{2}{\sqrt{5}}$. This tells us that $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$. **Find angles to two decimal places**. A calculator gives the answer $\theta \approx \boxed{26.57°}$. When we look at the diagram, 26.57° is a reasonable value for the angle between $3j$ and $2i + 4j$. To make sure you know how to use your calculator, use it to find $\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ and see if the result is approximately 26.57°.

<u>*Example exercise*</u>: Find the angle between $4i + j + 2k$ and $3j - 5k$.

Solution: Let $p = 4i + j + 2k$ and $q = 3j - 5k$. Thus j and $3j$ are like terms as are $2k$ and $-5k$. So $p \bullet q = 1(3) + 2(-5) = 3 - 10 = \underline{-7}$.

$|p|^2 = 4^2 + 1^2 + 2^2 = 16 + 1 + 4 = 21$. Thus $|p| = \underline{\sqrt{21}}$.

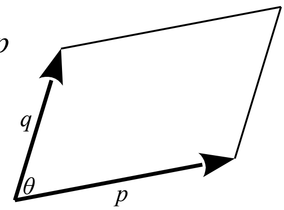
$|q|^2 = 3^2 + (-5)^2 = 9 + 25 = 34$. Thus $|q| = \underline{\sqrt{34}}$.

Thus $\cos\theta = \frac{-7}{\sqrt{21}\sqrt{34}} = \frac{-7}{\sqrt{714}}$ and $\theta = \cos^{-1}\left(\frac{-7}{\sqrt{714}}\right) \approx \boxed{105.19°}$.

Exercise 1: Find the angle between $2i - j + k$ and $3i + 4j$.

12) Parallelograms

Recall that the angle θ between the vectors p and q can be found using this formula:



$$\cos\theta = \frac{p \bullet q}{|p||q|}$$

We can rearrange the formula like this: $p \bullet q = |p||q| \cos \theta$. This formula shows that **if p and q are perpendicular, then $p \bullet q = 0$** since $\cos 90° = 0$. The vectors p and q shown above are two sides of a parallelogram. **The *area* of the parallelogram is $|p \times q|$.**

<u>*Example exercise*</u>: Find the area of the parallelogram formed by the vectors $3i - 4k$ and $2k$. Also find the angle between the vectors.

Solution:

$p = 3i - 4k$ and $q = 2k$	Give names to vectors
$(3i - 4k)(2k)$	Find 4-D product
$6ik - 8k^2$	Distribute
$6(-j) - 8(-1)$	$ik = -j$ and $k^2 = -1$
$8 - 6j$	Simplify and rearrange
$p \bullet q = \underline{-8}$ and $p \times q = \underline{-6j}$	Definitions of • and ×
Area $= \lvert -6j \rvert = \boxed{6}$	**Area = $\lvert p \times q \rvert$**
$\lvert p \rvert = 5$ and $\lvert q \rvert = 2$	$\lvert p \rvert^2 = 3^2 + (-4)^2$ and $\lvert q \rvert^2 = 2^2$
$\cos\theta = \frac{-8}{(5)(2)}$	$\boldsymbol{\cos\theta = \frac{p \bullet q}{\lvert p \rvert \lvert q \rvert}}$
$\cos\theta = -\frac{4}{5}$	Simplify
$\theta = \cos^{-1}\left(-\frac{4}{5}\right)$	Use inverse cosine
$\theta \approx \boxed{143.13°}$	Use calculator

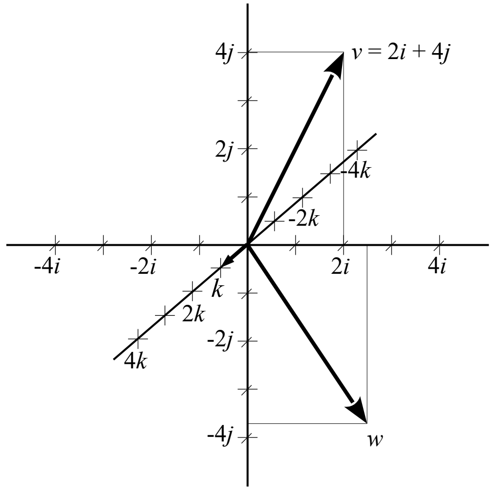
Make sure that your calculator gives the angle of 143.13° when you instruct it to find $\cos^{-1}\left(-\frac{4}{5}\right)$.

Exercise 1: Find the area of the parallelogram formed by the vectors $2i + j$ and $-j + k$. Also find the angle between the vectors.

Exercise 2: Write both the polar and rectangular forms of the product (3 cis 210°)(4 cis 120°). If you need assistance, study the first example in Lesson 5.

13) Vector Rotation

Suppose we want to rotate $2i + 4j$ about unit vector k by an angle of $-120°$. Call k the **rotation vector** and $-120°$ the **rotation angle**. To find the new vector, we multiply $(\cos -120° + k \sin -120°)$ and $(2i + 4j)$.



To use this method for rotating a vector, **the rotation vector needs to be a *unit vector* that is *perpendicular* to the vector being rotated.**

In general, to rotate vector v about unit vector u by an angle of θ, find the value of $(\cos\theta + u\sin\theta)\,v$. Remember, u must be a *unit vector* that is *perpendicular* to v (i.e., $|u| = 1$ and $u \cdot v = 0$). If the work is done correctly, *the resulting 4-D product should be a 3-D vector.*

<u>*Example exercise*</u>: Rotate $2i + 4j$ about k by an angle of $-120°$.

Solution: Obviously k is a unit vector perpendicular to $2i + 4j$. Thus we can use the formula $(\cos\theta + u\sin\theta)\,v$ where $u = \underline{k}$ and $v = \underline{2i + 4j}$ and $\theta = \underline{-120°}$. By studying the unit circle, we observe that cis $-120°$ = cis $240°$ since $-120° + 360° = 240°$. Thus we see that $\text{cis } -120° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. Recall that **cis $\boldsymbol{\theta} = \cos\boldsymbol{\theta} + \boldsymbol{i}\sin\boldsymbol{\theta}$.** Thus $\cos -120° = -\frac{1}{2}$ and $\sin -120° = -\frac{\sqrt{3}}{2}$. We can now evaluate the expression $(\cos\theta + u\sin\theta)\,v$:

$(\cos -120° + k \sin -120°)(2i + 4j)$	Substitute
$\left(-\frac{1}{2} + -\frac{\sqrt{3}}{2}k\right)(2i + 4j)$	See unit circle
$-i - 2j - \sqrt{3}\,ki - 2\sqrt{3}\,kj$	Distribute
$-i - 2j - \sqrt{3}\,j + 2\sqrt{3}\,i$	$ki = j$ and $kj = -i$
$\boxed{(-1+2\sqrt{3})i+(-2-\sqrt{3})j}$	Rearrange

That vector is approximately $2.46i - 3.73j$ and is vector w in the diagram above. Observe that $|w| = |v|$.

<u>*Example exercise*</u>: Rotate $4i + 3k$ about $\frac{3}{5}i - \frac{4}{5}k$ by 90°.

Solution: Since $(\frac{3}{5}i - \frac{4}{5}k) \bullet (4i + 3k) = \frac{3}{5}(4) - \frac{4}{5}(3) = \frac{12}{5} - \frac{12}{5} = 0$, the rotation vector $\frac{3}{5}i - \frac{4}{5}k$ is perpendicular to $4i + 3k$. We also see that $\left|\frac{3}{5}i - \frac{4}{5}k\right| = 1$ since $\left|\frac{3}{5}i - \frac{4}{5}k\right|^2 = \left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1$. Thus we can let $u = \underline{\frac{3}{5}i - \frac{4}{5}k}$ and $v = \underline{4i + 3k}$ and $\theta = \underline{90°}$ in the expression $(\cos\theta + u\sin\theta)\,v$:

$[\cos 90° + (\frac{3}{5}i - \frac{4}{5}k)\sin 90°](4i + 3k)$	Substitute
$[0 + (\frac{3}{5}i - \frac{4}{5}k)(1)](4i + 3k)$	See unit circle
$(\frac{3}{5}i - \frac{4}{5}k)(4i + 3k)$	Simplify
$\frac{12}{5}i^2 + \frac{9}{5}ik - \frac{16}{5}ki - \frac{12}{5}k^2$	Distribute
$-\frac{12}{5} - \frac{9}{5}j - \frac{16}{5}j + \frac{12}{5}$	$i^2 = k^2 = -1$ $ik = -j$ and $ki = j$
$\boxed{-5j}$	Simplify

The following paragraph is a simple cryptogram that introduces terminology not needed for these lessons. However, people often use that terminology when talking about 2-D and 4-D numbers.

Uif gpvs-ejnfotjpobm ovncfst jo uiftf mfttpot bsf *rvbufsojpot*. Uxp-ejnfotjpobm ovncfst bsf dbmmfe *dpnqmfy ovncfst*. Ovncfst uibu dpoubjo uif trvbsf sppu pg ofhbujwf pof bsf vogpsuvobufmz dbmmfe *jnbhjobsz ovncfst*. Ep opu mfu tvdi ufsnjopmphz jogmvfodf zpv up uijol uibu uiftf ovncfst bsf tpnfipx mftt sfbm uibo ofhbujwf ovncfst.

Both 2-D numbers and 4-D numbers are powerful tools. They help us who live in 3-D space. These numbers enrich our lives. Expand your mind – think outside of the number line!

Exercise 1: Rotate $8i + 2k$ about $-j$ by 60°. In the regular exercises, the rotation vector is always a unit vector perpendicular to the vector being rotated.

Extra credit 1: If the vector v being rotated about the unit vector u is *not* perpendicular to u, then the following formula can be used:

$$\left(\cos\frac{\theta}{2} + u\sin\frac{\theta}{2}\right)v\left(\cos\frac{\theta}{2} - u\sin\frac{\theta}{2}\right)$$

Use that formula to rotate $2i + 4j$ about j by 60°.

14) Squaring 2-D Numbers

Consider the expression z^2. If we substitute the number i for z in that expression, we get i^2 which is –1. Now substitute –1 for z in z^2. The result is $(-1)^2 = 1$. Now substitute 1 for z in z^2. We get $1^2 = 1$. Consider the table to the right. We let z_0 represent the initial value of z. In this case, $z_0 = i$. Then $z_1 = i^2 = -1$. Then $z_2 = (-1)^2 = 1$, etc. Each number below z_0 is the square of the number above it.

z_0	$= i$
z_1	$= -1$
z_2	$= 1$
z_3	$= 1$
z_4	$= 1$

The table below lists the results when we start with various values of z_0. Each number below z_0 is the square of the number above it. The first column begins with $z_0 = i$ as discussed above. In that column, the absolute value of each number is 1.

The second column begins with $z_0 = 2i$. For that column, the absolute values *increase* as we go down the column. The third column begins with $z_0 = 1.1i$. In that column, the absolute values also *increase*, but initially not as fast as they do when $z_0 = 2i$. In this table, if a number has more than 4 decimal places, then it is rounded to the 4th decimal place.

The fourth column begins with $z_0 = 0.9i$. In that column, the absolute values *decrease* as we go down the column. The fifth column begins with z_0 = cis 30°. When multiplying 2-D numbers, the angles are added. Thus (cis 30°)^2 = (cis 30°)(cis 30°) = cis 60°.

z_0	i	$2i$	$1.1i$	$0.9i$	cis 30°	$1 + i$
z_1	–1	–4	–1.21	–0.81	cis 60°	$2i$
z_2	1	16	1.4641	0.6561	cis 120°	–4
z_3	1	256	2.1436	0.4305	cis 240°	16
z_4	1	65,536	4.5950	0.1853	cis 120°	256

Observe that (cis 240°)^2 = (cis 240°)(cis 240°) = cis 480°. Then since 480° – 360° = 120°, we conclude that (cis 240°)^2 = cis 120°. In this column, the numbers stay on the unit circle. Thus each absolute value is 1.

We substitute z_0 for z in the expression z^2 to get z_1. Then we substitute z_1 for z in the expression z^2 to get z_2. Then we substitute z_2 for

z in the expression z^2 to get z_3, etc. When we do this type of substitution, we say that we are **iterating** z^2. We take the result produced by evaluating the expression for a certain value and substitute that result back into the expression to get a new result, etc.

Recall that iterating z^2 when $z_0 = 2i$ produced numbers whose absolute values continually increased. If we continue to iterate z^2, the sizes of the numbers will increase without limit. We will call the list of numbers $\{2i, -4, 16, 256, \ldots\}$ the **orbit** for $z_0 = 2i$. In that orbit, the absolute values increase without limit. Thus we say that the orbit for $z_0 = 2i$ is **attracted to infinity**.

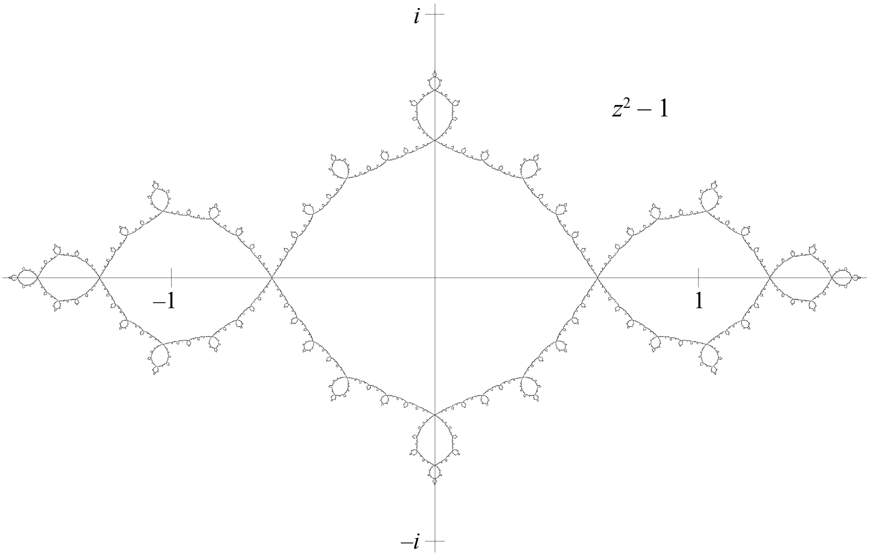
When $z_0 = i$, the orbit is $\{i, -1, 1, 1, 1, \ldots\}$. That orbit is not attracted to infinity. When $z_0 = 0.9i$, the orbit is $\{0.9i, -0.81, 0.6561, 0.4305, 0.1853, \ldots\}$. Iterating z^2 results in the numbers of the orbit getting closer and closer to 0. Thus the orbit for $z_0 = 0.9i$ is not attracted to infinity.

The expression z^2 divides the 2-D number plane into two regions – numbers whose orbits are attracted to infinity and numbers whose orbits are not attracted to infinity. Numbers *outside* the unit circle produce orbits attracted to infinity. Numbers either *on* the unit circle or *inside* the unit circle produce orbits that are not attracted to infinity.

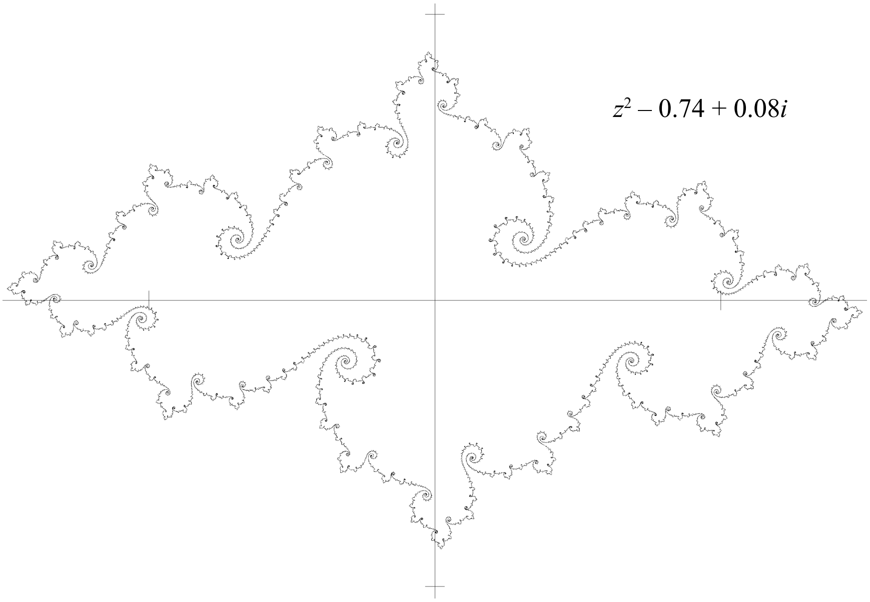
The numbers on the unit circle form the *boundary* of the region of the numbers whose orbits are attracted to infinity. Because of the work of the man Gaston Julia, we call the unit circle the **Julia set** for the expression z^2.

The diagram at the top of the next page attempts to show the Julia set for the expression $\mathbf{z^2 - 1}$. The design shows the approximate location of the boundary of the region of the numbers whose orbits are attracted to infinity as $z^2 - 1$ is iterated.

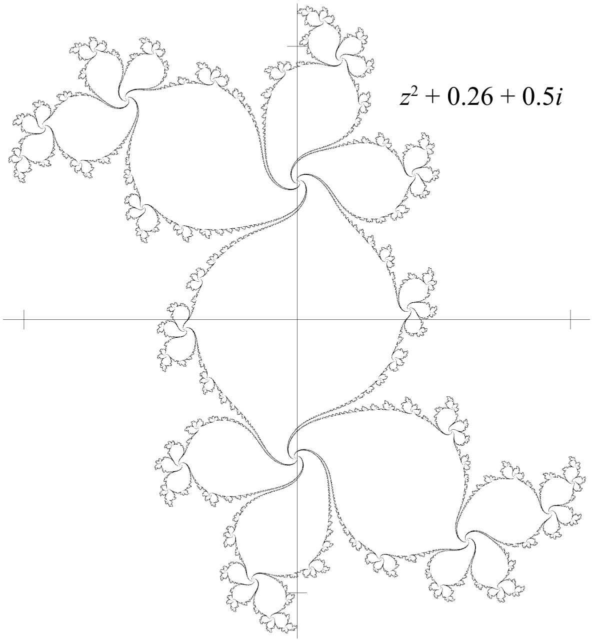
When $z_0 = i$, then $z_1 = i^2 - 1 = -1 - 1 = \underline{-2}$. So $z_2 = (-2)^2 - 1 = \underline{3}$. Then $z_3 = 3^2 - 1 = \underline{8}$. The orbit is $\{i, -2, 3, 8, 63, 3968, \ldots\}$. These calculations show that the orbit is attracted to infinity. From the diagram, we can see that i is outside of the design. The 2-D numbers inside the design produce orbits that are not attracted to infinity. The numbers outside of the design produce orbits attracted to infinity.



The design below shows the approximate location of the Julia set for the expression $\boldsymbol{z^2 - 0.74 + 0.08i}$. On this diagram, the locations of 1, –1, *i*, and –*i* are indicated using tick marks without labels. The numbers outside the design produce orbits attracted to infinity. The numbers inside the design produce orbits that are not attracted to infinity. The numbers on the Julia set are boundary points that produce orbits that stay on the Julia set.



The design to the right is the Julia set for the expression $\mathbf{z^2 + 0.26 + 0.5i}$. Notice that the number $1 + 0i$ is outside of the design. This indicates that $z_0 = 1$ produces an orbit attracted to infinity.



Example exercise: For $z^2 + 0.26 + 0.5i$, find z_1 and z_2 when $z_0 = 1$. Round decimals to the nearest hundredth.

Solution:

$z_1 = 1^2 + 0.26 + 0.5i$	$z_0 = 1$
$= \boxed{1.26 + 0.5i}$	Simplify
$z_2 = (1.26 + 0.5i)^2 + 0.26 + 0.5i$	$z_1 = 1.26 + 0.5i$
$= (1.26 + 0.5i)(1.26 + 0.5i) + 0.26 + 0.5i$	Expand
$\approx 1.59 + 0.63i + 0.63i - 0.25 + 0.26 + 0.5i$	Distribute
$\approx \boxed{1.60 + 1.76i}$	Simplify

Example exercise: Using the values of z_1 and z_2 from above, evaluate $|z_1|$ and $|z_2|$ to the nearest hundredth.

Solution: $|z_1|^2 = 1.26^2 + 0.5^2 \approx 1.84$. Thus $|z_1| \approx \boxed{1.36}$.

$|z_2|^2 \approx 1.60^2 + 1.76^2 \approx 5.66$. Thus $|z_2| \approx \boxed{2.38}$.

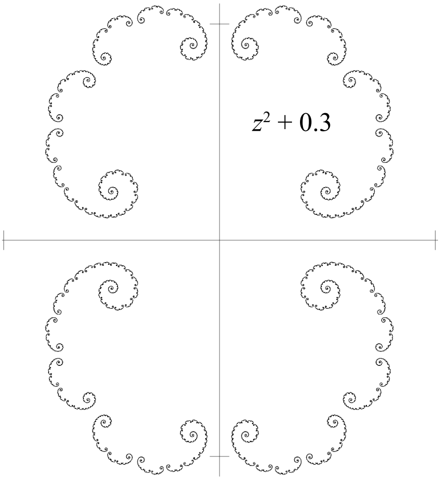
Exercise 1: For the expression $z^2 + 0.26 + 0.5i$, find z_1 and z_2 when $z_0 = i$. Round decimals to the nearest hundredth.

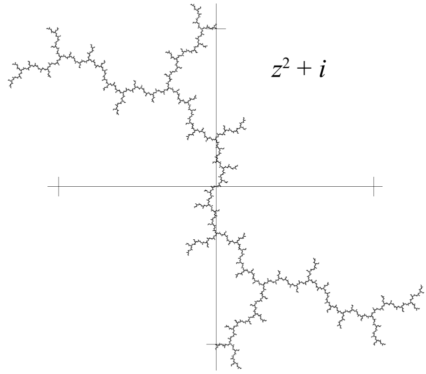
Exercise 2: Using z_1 and z_2 from *Exercise 1*, find $|z_1|$ and $|z_2|$ to the nearest hundredth.

Consider these pairs of letters: ab, bc, cd, de, etc. They may help you decode a cryptogram.

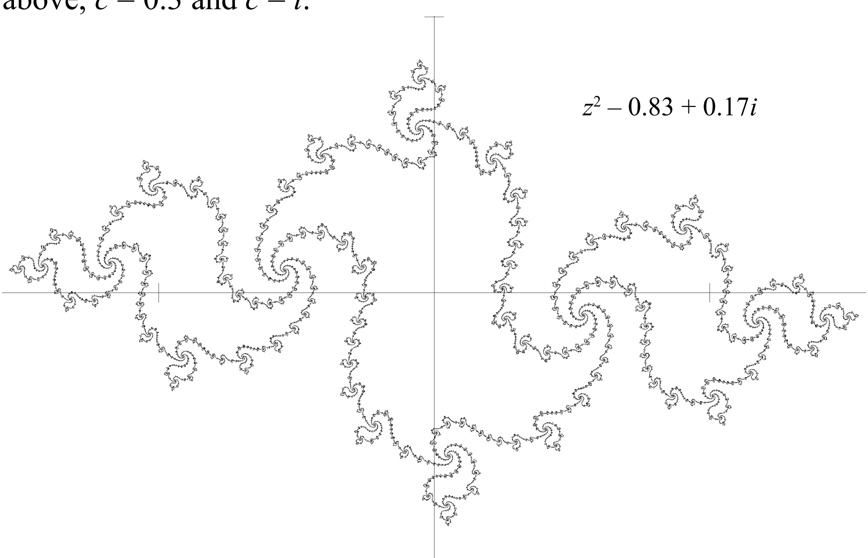
15)

Julia Sets





The diagram to the above-left shows the Julia set for $z^2 + 0.3$. Observe that this Julia set consists of disconnected pieces. The diagram to the above-right shows the Julia set for $z^2 + i$. This Julia set is **connected**. The Julia set for $z^2 + 0.3$ is **disconnected**. The Julia sets displayed in the previous lesson were connected sets like the Julia set for $z^2 – 0.83 + 0.17i$ shown below. All of the Julia sets discussed in these lessons are for expressions of the form $\boldsymbol{z^2 + c}$. For the Julia set below, $c = –0.83 + 0.17i$. For the Julia sets displayed above, $c = 0.3$ and $c = i$.



Five of the designs on the title page are Julia sets. For the top two designs, $c = -0.0986 - 0.6522i$ and $c = -0.0979 + 0.6522i$. The numbers $-0.14495 + 0.651i$ and $-0.14558 - 0.651i$ are the values of c for the bottom two designs. The central design shows the Julia set for $c = 0.26$ after the Julia set has been rotated 90°.

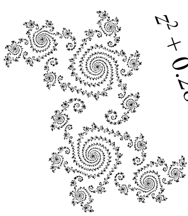
<u>*Example exercise*</u>: Find z_1 and z_2 when $c = i$ and $z_0 = 1 - i$.

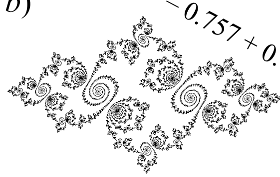
Solution: We substitute into the expression $\boldsymbol{z^2 + c}$:

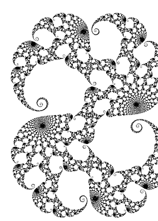
$z_1 = (1 - i)^2 + i$	$z_0 = 1 - i$ and $c = i$
$= (1 - i)(1 - i) + i$	Expand
$= 1 - i - i - 1 + i$	Distribute
$= \boxed{-i}$	Simplify
$z_2 = (-i)^2 + i$	$z_1 = -i$ and $c = i$
$= \boxed{-1 + i}$	Simplify

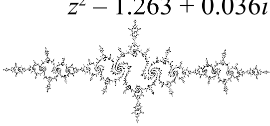
Exercise 1: For $z^2 + c$, find z_1 and z_2 when $c = -i$ and $z_0 = i$.

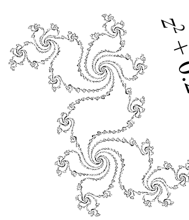
Exercise 2: For each of the following Julia sets, write *connected* or *disconnected*.

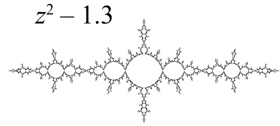
a)

$z^2 + 0.28 + 0.48i$

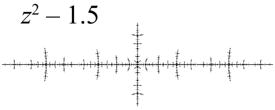
b)

$z^2 - 0.757 + 0.071i$

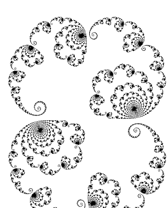
c)

$z^2 + 0.26274 + 0.00225i$

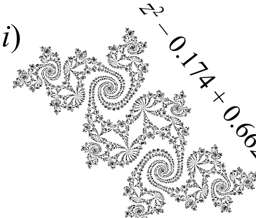
d)
$z^2 - 1.263 + 0.036i$


e)

$z^2 + 0.28 + 0.49i$

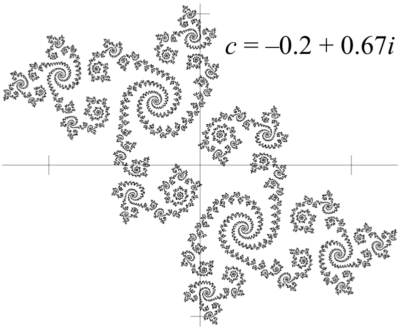
f)
$z^2 - 1.3$


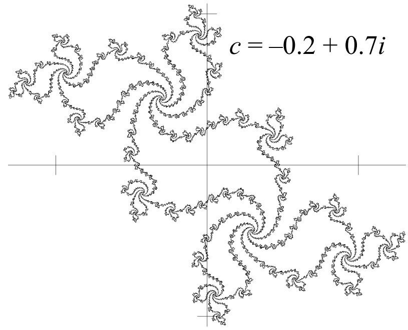
g)
$z^2 - 1.5$


h)

$z^2 + 0.2627 + 0.002i$

i)

$z^2 - 0.174 + 0.662i$

16) The Expression $z^2 + c$





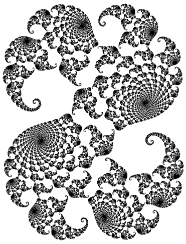
As the diagrams above show, the Julia set for $z^2 - 0.2 + 0.67i$ is disconnected and the Julia set for $z^2 - 0.2 + 0.7i$ is connected. To decide if a Julia set is connected, we can **analyze the orbit when $z_0 = 0$**. *If the orbit for $z_0 = 0$ is **attracted to infinity**, then the Julia set is **disconnected**. If that orbit is **not attracted to infinity**, then the Julia set is **connected**.*

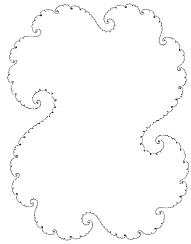
Consider $z^2 - 0.2 + 0.67i$. The table to the right shows some values for that orbit when $z_0 = 0$. By the time the orbit reaches z_{26}, it is obvious that the orbit is *attracted to infinity*. Thus the Julia set is *disconnected*. **In an orbit for $z^2 + c$ where $|c| \le 2$, if the size of a number ever exceeds 2, then the orbit is attracted to infinity**. FOR EACH JULIA SET IN THESE LESSONS, $|c| \le 2$.

z_0	$=$	0
z_1	$=$	$-0.2 + 0.67i$
z_2	$\approx$	$-0.61 + 0.40i$
z_3	$\approx$	$0.01 + 0.18i$
	$\vdots$	
z_{25}	$\approx$	$-1.97 + 3.80i$
z_{26}	$\approx$	$-10.79 - 14.33i$

Consider $z^2 - 0.2 + 0.7i$. The table to the right shows some of the values a computer calculated for the orbit when $z_0 = 0$. Observe that $z_{204} \approx z_{201}$. From the numbers in the table, it may appear that $z_{204} = z_{201}$. However, the values shown for z_{204} and z_{201} are rounded. Thus $z_{204} \approx z_{201}$. Also, $z_{205} \approx z_{202}$ and $z_{206} \approx z_{203}$. If the iteration process is continued for the expression $z^2 - 0.2 + 0.7i$, the approximate values in the orbit will continue to be $0.13 + 0.14i$, $-0.20 + 0.74i$, and $-0.70 + 0.40i$. Since the orbit stays near these numbers forever, the orbit is *not attracted to infinity* and the Julia set is *connected*.

z_0	$=$	0
z_1	$=$	$-0.2 + 0.7i$
z_2	$=$	$-0.65 + 0.42i$
z_3	$\approx$	$0.05 + 0.15i$
	$\vdots$	
z_{201}	$\approx$	$0.13 + 0.14i$
z_{202}	$\approx$	$-0.20 + 0.74i$
z_{203}	$\approx$	$-0.70 + 0.40i$
z_{204}	$\approx$	$0.13 + 0.14i$
z_{205}	$\approx$	$-0.20 + 0.74i$
z_{206}	$\approx$	$-0.70 + 0.40i$

A) 

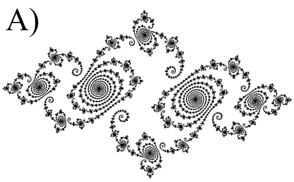
B) 

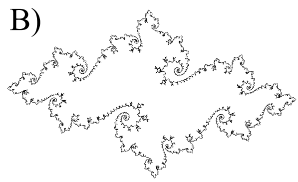
z_{95}	≈	$0.44 + 0.19i$
z_{96}	≈	$0.44 + 0.18i$
z_{97}	≈	$0.44 + 0.18i$
z_{98}	≈	$0.44 + 0.18i$
z_{99}	≈	$0.44 + 0.18i$

Example exercise: Shown above is part of the orbit for $z^2 + c$ when $c = 0.28 + 0.02i$ and $z_0 = 0$. Find $|z_{99}|$ to the nearest hundredth. Which Julia set above was produced using $c = 0.28 + 0.02i$?

Solution: $|z_{99}|^2 \approx 0.44^2 + 0.18^2 \approx 0.23$. Thus $\boxed{|z_{99}| \approx 0.48}$.

The values in the orbit are staying near $0.44 + 0.18i$. Thus the orbit is *not attracted to infinity* and the Julia set is *connected*. Therefore the answer is $\boxed{\text{B}}$ since it is connected and A is disconnected.

A) 

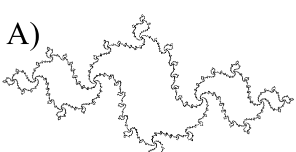
B) 

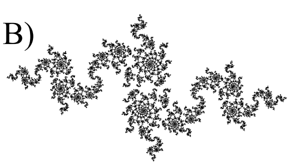
z_{25}	≈	$-0.36 + 0.69i$
z_{26}	≈	$-1.09 - 0.39i$
z_{27}	≈	$0.28 + 0.96i$
z_{28}	≈	$-1.58 + 0.65i$
z_{29}	≈	$1.33 - 1.95i$

Example exercise: Shown above is part of the orbit for $z^2 + c$ when $c = -0.75 + 0.11i$ and $z_0 = 0$. Find $|z_{29}|$ to the nearest hundredth. Which Julia set above was produced using $c = -0.75 + 0.11i$?

Solution: $|z_{29}|^2 \approx 1.33^2 + (-1.95)^2 \approx 5.57$. Thus $\boxed{|z_{29}| \approx 2.36}$.

Since $|z_{29}| > 2$, the orbit is *attracted to infinity* and the Julia set is *disconnected*. Thus the answer is $\boxed{\text{A}}$ since it is disconnected and B is connected.

A) 

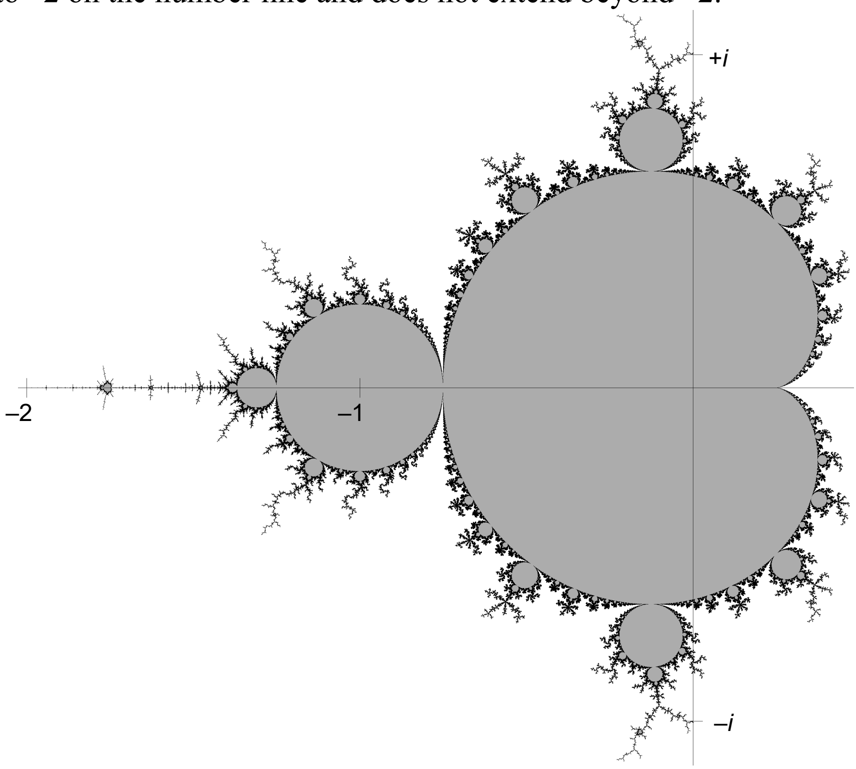
B) 

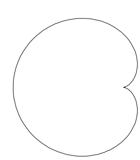
z_{74}	≈	$-0.05 - 0.23i$
z_{75}	≈	$-0.95 + 0.22i$
z_{76}	≈	$-0.05 - 0.22i$
z_{77}	≈	$-0.95 + 0.22i$
z_{78}	≈	$-0.05 - 0.22i$

Exercise 1: Shown above is part of the orbit for $z^2 + c$ when $z_0 = 0$ and $c = -0.9 + 0.2i$. Find $|z_{78}|$ to the nearest hundredth. Which Julia set above was produced using $c = -0.9 + 0.2i$? (In these types of exercises, the z's in the orbits have been selected so that you can determine if the set is connected by analyzing the absolute value of the last z displayed. If that absolute value is not greater than 2, then the orbit is not attracted to infinity.)

17) Mapping Connected Julia Sets

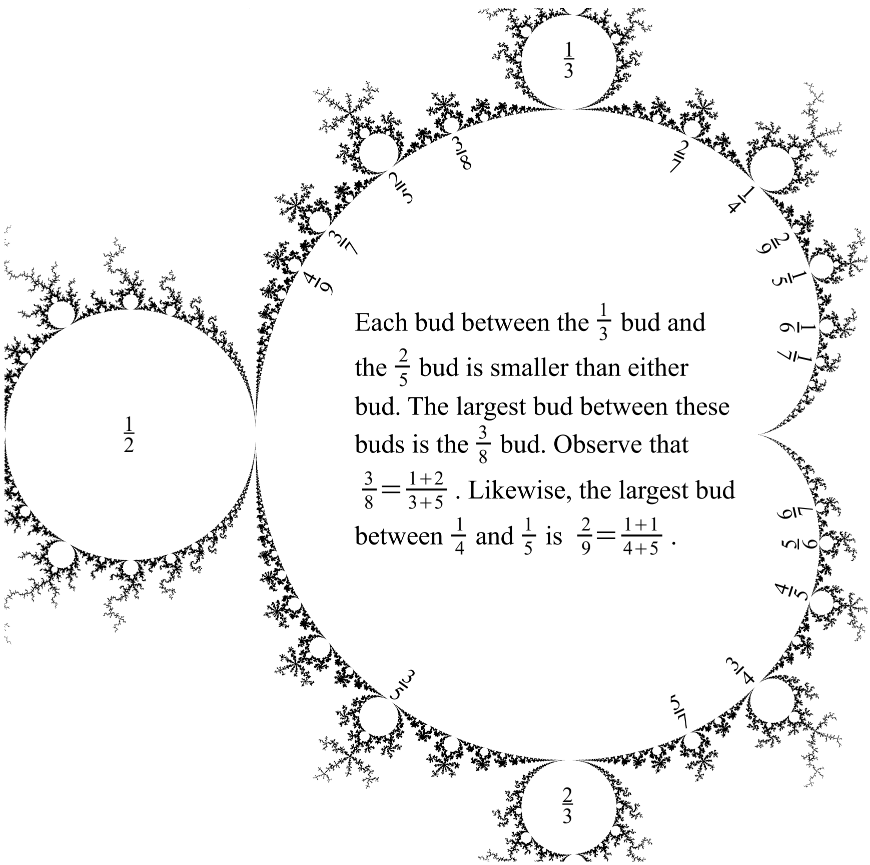
Recall that for $z^2 + c$, some values of c produce connected Julia sets and some values produce disconnected Julia sets. **A Julia set for $z^2 + c$ is *connected* if and only if the orbit for $z_0 = 0$ is *not* attracted to infinity**. Whenever $|c| > 2$, the orbit for $z_0 = 0$ is attracted to infinity and the Julia set is disconnected. If $|c| \leq 2$, then the Julia set might be connected or disconnected. The diagram below shades in the values of c that produce connected Julia sets. The boundary of the design is emphasized with black. It may not be obvious that there is a spike on the left side of the design that extends to –2 on the number line and does not extend beyond –2.





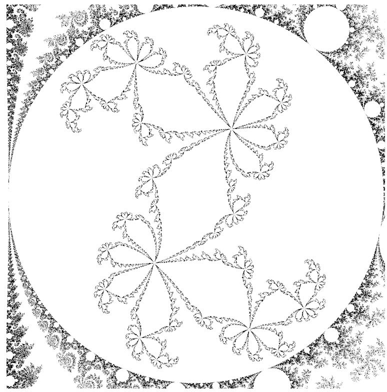
This design is a map showing the locations of the values of c that produce connected Julia sets. This design is called the **Mandelbrot set** after Benoit B. Mandelbrot. The largest region in the design is shaped like a sideways heart and is called a *cardioid* because of its shape. An outline of a cardioid is shown to the left.

Attached to the main cardioid in the Mandelbrot set are buds. Attached to these buds are smaller buds. Attached to these buds are even smaller buds, etc. We will focus on the buds attached directly to the main cardioid. There are an infinite number of these buds. Thankfully, there is an easy way to label these buds. We will use *fractions* for the labels. Below is a magnified view of *part* of the Mandelbrot set showing the main cardioid and buds on it. Only the boundary of the Mandelbrot set is shaded. Most labels are in the cardioid because of the small sizes of the buds.



The equation $\boldsymbol{c=\frac{1}{2}\operatorname{cis}\theta-\frac{1}{4}\operatorname{cis}2\theta}$ calculates values of c on the main cardioid. This equation shows that the cardioid can be formed by two connected rotating rods where the second rod initially points to the left. Also, the second rod rotates twice as fast as the first rod and is half as long as the first rod. To find the value of c

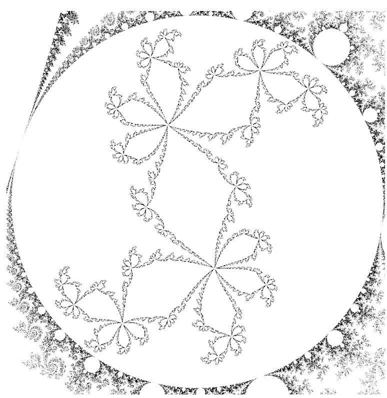
where the $\frac{2}{3}$ bud attaches to the main cardioid, substitute $\frac{2}{3}\cdot 360°$ for θ in the equation $\boldsymbol{c=\frac{1}{2}\operatorname{cis}\theta-\frac{1}{4}\operatorname{cis}2\theta}$. Similarly, to find the value of c where the $\frac{5}{7}$ bud attaches to the main cardioid, substitute $\frac{5}{7}\cdot 360°$ for θ in that equation. The pattern continues.



<u>*Example exercise*</u>: A magnified view of the $\frac{1}{6}$ bud is shown to the right. Inside the bud is shown the Julia set for $c = 0.389 + 0.217i$ which is near the center of the bud. On the left of this bud is where it attaches to the main cardioid. Find the *exact* value of c where the $\frac{1}{6}$ bud attaches to the main cardioid.

Solution:

$\frac{1}{2}\operatorname{cis}60°-\frac{1}{4}\operatorname{cis}120°$	$\theta=\frac{1}{6}\cdot 360° = 60°$ and $2\theta = 2\cdot 60° = 120°$
$\frac{1}{2}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)-\frac{1}{4}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$	See unit circle
$\frac{1}{4}+\frac{\sqrt{3}}{4}i+\frac{1}{8}-\frac{\sqrt{3}}{8}i$	Distribute
$\boxed{\frac{3}{8}+\frac{\sqrt{3}}{8}i}$	Simplify

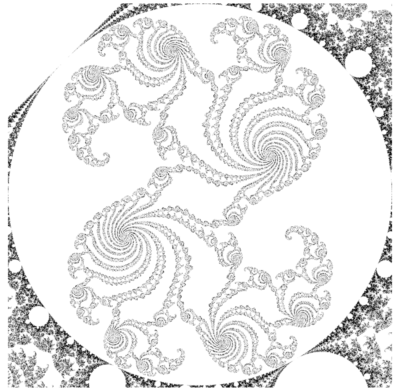


<u>*Example exercise*</u>: A magnified view of the $\frac{4}{5}$ bud is shown to the right. Inside the bud is shown the Julia set for $c = 0.38 - 0.335i$ which is near the center of the bud. Near the left of the bud is where it attaches to the cardioid. To the *nearest hundredth*, find the value of c where the $\frac{4}{5}$ bud attaches to the main cardioid.

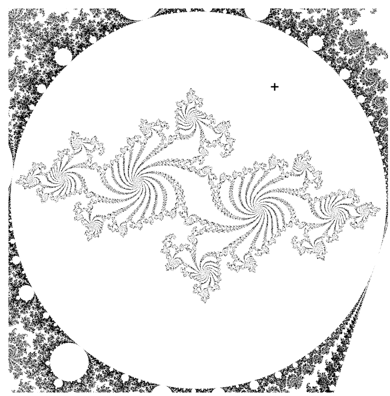
Solution:

$\frac{1}{2}\operatorname{cis}288°-\frac{1}{4}\operatorname{cis}576°$	$\theta=\frac{4}{5}\cdot 360° = 288°$
$\frac{1}{2}\cos 288°+\frac{1}{2}i\sin 288°-\frac{1}{4}\cos 576°-\frac{1}{4}i\sin 576°$	Expand cis
$\frac{1}{2}\cos 288°-\frac{1}{4}\cos 576°+i\left(\frac{1}{2}\sin 288°-\frac{1}{4}\sin 576°\right)$	Rearrange
$\boxed{0.36-0.33i}$	Simplify

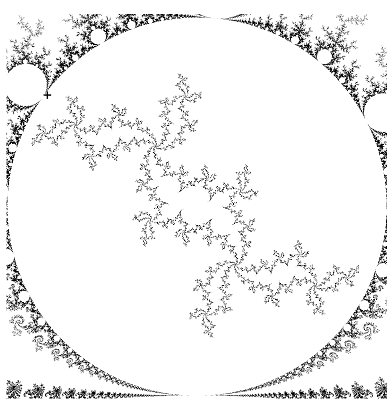
Exercise 1: A magnified view of the $\frac{1}{12}$ bud is displayed to the right. Inside the bud is shown the Julia set for $c = 0.3076 + 0.032i$ which is inside the bud near its left side. At the upper-left of the bud is where it attaches to the cardioid. Find the *exact* value of c where the $\frac{1}{12}$ bud attaches to the main cardioid.



Exercise 2: A magnified view of the $\frac{7}{15}$ bud is shown to the right. Inside the bud is shown the Julia set for $c = -0.72 + 0.21i$ which is the location marked with a plus sign. Near the right of the bud is where it attaches to the cardioid. To the *nearest hundredth*, find the value of c where the $\frac{7}{15}$ bud attaches to the main cardioid.

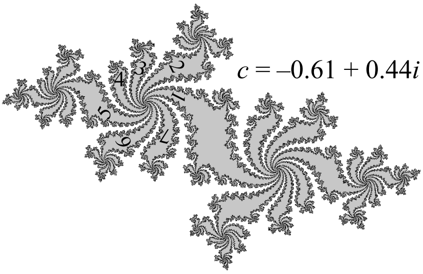


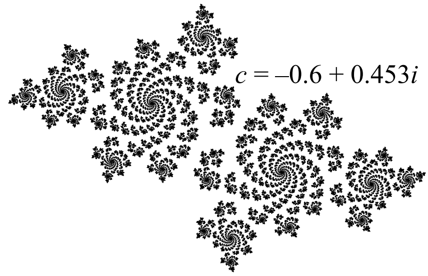
Exercise 3: A magnified view of the $\frac{1}{3}$ bud is shown to the right. Inside the bud is shown the Julia set for $c = -0.2 + 0.8i$ which is the location marked with a plus sign. At the bottom of the bud is where it attaches to the cardioid. Find the *exact* value of c where the $\frac{1}{3}$ bud attaches to the main cardioid.



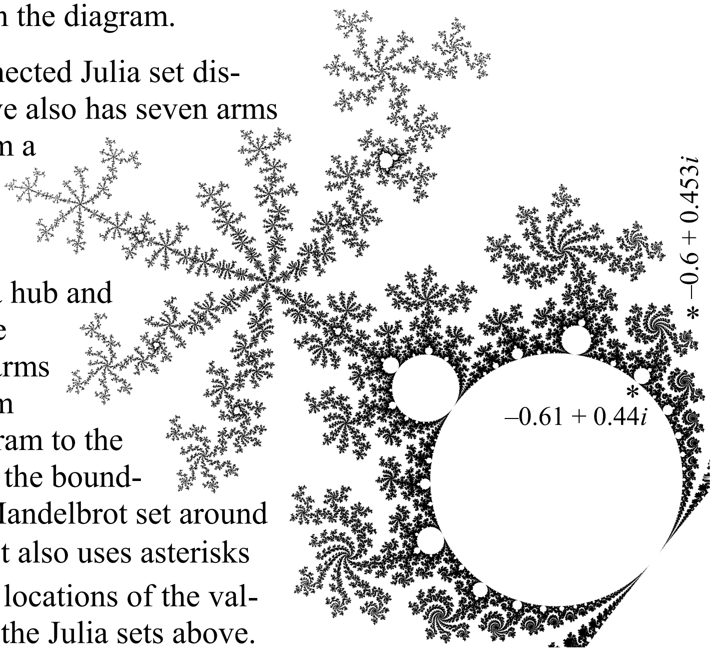
Tlw rh zdvhlnv. Sv wrivxgvw nv rm wvevolkrmt gsvhv ovhhlmh rm hkrgv lu nrhgzpvh R nzwv. Nzb Sv yv tolirurvw yb lfi hgfwb lu Srh drhwln. Gdl-wrnvmhrlmzo zmw ulfi-wrnvmhrlmzo mfnyvih svok fh fmwvihgzmw dszg Sv szh nzwv. Uli zm vovxgilmrx xlkb lu gsvhv ovhhlmh, hvmw z ivjfvhg gl qlsmist zg tnzro wlg xln. Gszg rh z ezorw zwwivhh zh lu gsv bvzi NNCERR.

48) Bud Denominators

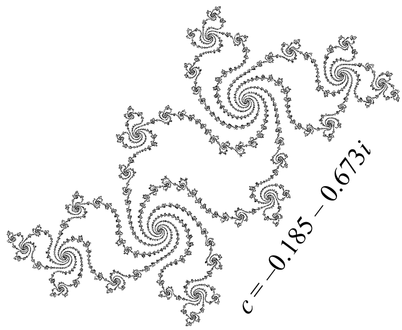




The Julia set for *c* = –0.61 + 0.44*i* is connected. It also encloses regions which are shaded in the diagram above. *A connected Julia set with all enclosed regions is a **filled-in Julia set***. For the filled-in Julia set displayed above, seven **arms** coming from a **hub** are numbered in the diagram.

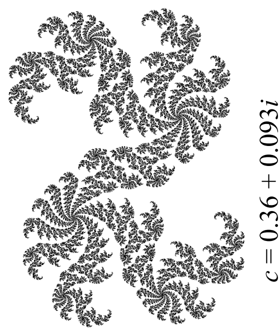


The disconnected Julia set displayed above also has seven arms coming from a hub. Prove this to yourself by finding a hub and counting the number of arms coming from it. The diagram to the right shows the boundary of the Mandelbrot set around the $\frac{3}{7}$ bud. It also uses asterisks to show the locations of the values of *c* for the Julia sets above. The value of *c* for the connected Julia set is in the $\frac{3}{7}$ bud and thus in the Mandelbrot set. If the value of *c* for a Julia set is in a bud attached to the main cardioid, then the **number of arms** coming from a hub is the **denominator** of the fraction for the bud. The value of *c* for the disconnected Julia set is outside of the Mandelbrot set. However, it is near enough to the $\frac{3}{7}$ bud that the disconnected Julia set has seven arms coming from a hub. If the value of *c* for a disconnected Julia set is near a bud attached to the main cardioid, then the number of arms coming from a hub is the denominator of the fraction for the bud.



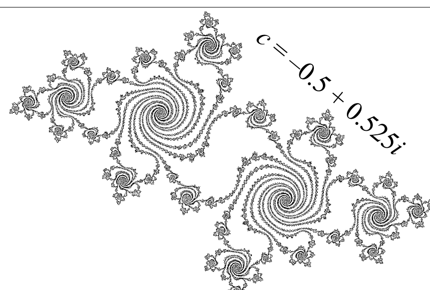
Example exercise: The value of c for the Julia set to the right is in a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?

Solution: If this connected Julia set were filled-in, then three filled-in arms would come from each hub. Thus the denominator is $\boxed{3}$.

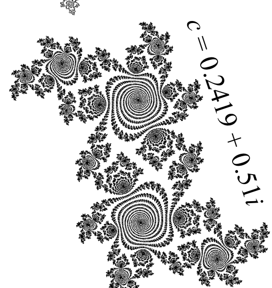


Example exercise: The value of c for the Julia set to the right is near a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?

Solution: Careful counting shows that eight arms come from a hub. Thus the denominator is $\boxed{8}$.

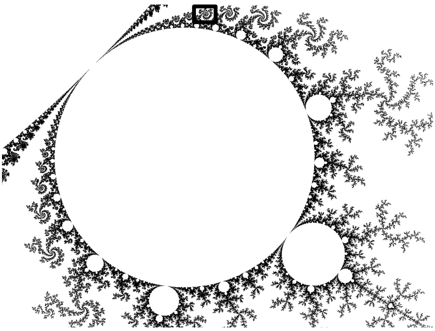


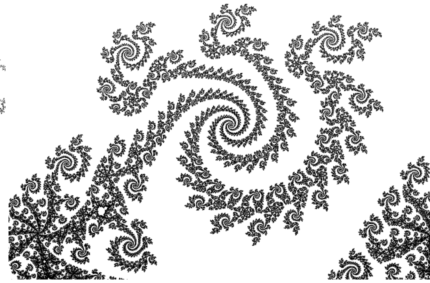
Exercise 1: The value of c for the Julia set to the right is in a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?



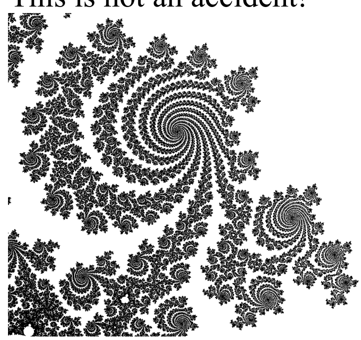
Exercise 2: The value of c for the Julia set to the right is near a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?

19) Mandelbrot Spirals



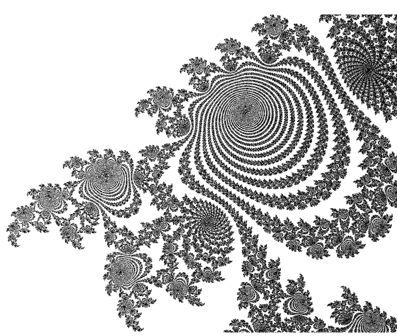


To the above-left is a magnified view of the $\frac{3}{4}$ bud. Look closely at the top of the bud and notice the small rectangle. If that part of the boundary of the Mandelbrot set is magnified, we get the spiral design shown to the above-right. The hub at the center of the main spiral in the spiral design has four arms coming from it. Count them yourself. Observe that *four* is the denominator of $\frac{3}{4}$ – the label for the bud near this spiral design. This is not an accident!



Example exercise: The spiral to the right is near a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?

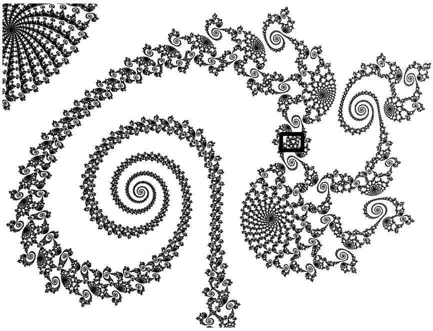
Solution: Carefully count the number of arms coming from the center of the main spiral. The answer is 9.

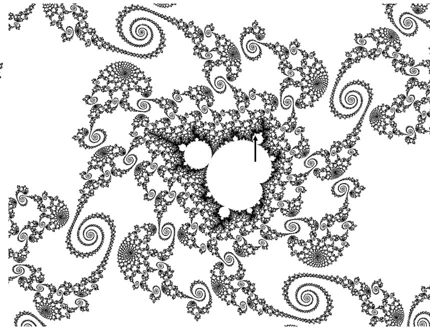


Exercise 1: The spiral to the right is near a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud? Do not allow the smaller spiral in the upper-right corner to confuse you. The main spiral gives the information you need.

Consider these pairs of letters: az, by, cx, dw, etc. They may help you decode a cryptogram.

20) Mandelbrot Similarity

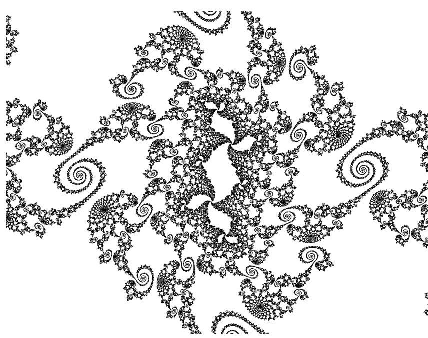


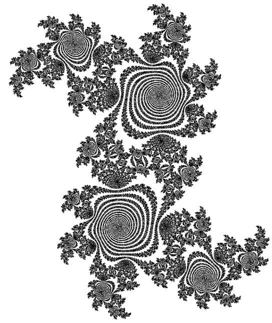


Shown to the above-left is a magnified view of a spiral near the $\frac{1}{2}$ bud. The main spiral in that design has 2 arms coming from the center. Find the small rectangle that has been drawn on the design. If we magnify the region in the rectangle, we get the design to the above-right. That design contains within it a shape that looks a lot like the Mandelbrot set. This shape is a **miniature Mandelbrot**. Remember that this design is a small part of the boundary of the Mandelbrot set. The Mandelbrot set contains within it an infinite number of small shapes that look a lot like the entire Mandelbrot set. Thus the Mandelbrot set is **approximately self-similar**.

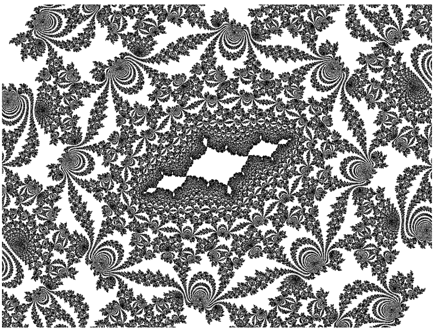
In the miniature Mandelbrot above an arrow points into the $\frac{1}{3}$ bud. The tip of that arrow is near $c = -0.761473 + 0.089919i$. The connected Julia set to the right is for that value of c. Since that value is near the $\frac{1}{2}$ bud of the main Mandelbrot set, the main spirals have 2 arms.

If we greatly magnify the very center of the Julia set, we get the design to the right. Since the value of c is in the $\frac{1}{3}$ bud of a miniature Mandelbrot, the design at the center has 3 arms coming from each hub.



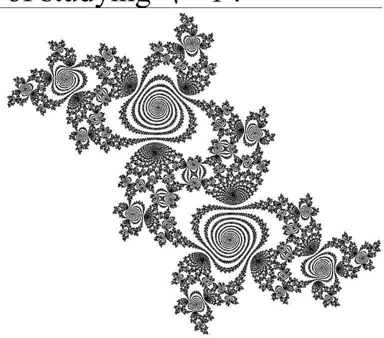


<u>*Example exercise*</u>: The value of *c* for the connected Julia set to the right is from a bud in a miniature Mandelbrot. The design below the Julia set is a greatly magnified view of the center of this Julia set. For the Julia set to the right, $c = 0.355414302 + 0.33512317i$. Write the denominator for the bud of the *main* Mandelbrot set that this value is near. Also write the denominator for the bud of the *miniature* Mandelbrot from which this value comes.

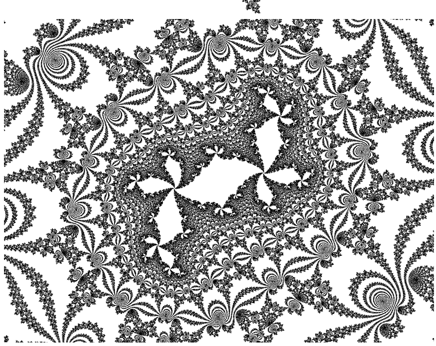


Solution: Each *main* spiral in the Julia set has 5 arms. The design in the center of the *magnified* view has 2 arms from each hub. Thus the denominators are $\boxed{5}$ and $\boxed{2}$.

What amazing designs we discover as we study how the simple expression $z^2 + c$ behaves in the 2-D number plane! We should praise God for giving us the privilege of studying $\sqrt{-1}$.



Exercise 1: The value of *c* for the connected Julia set to the right is from a bud in a miniature Mandelbrot. The design below the Julia set is a greatly magnified view of the center of this Julia set. For the Julia set to the right, $c = -0.103125475 + 0.651512577i$. Write the denominator for the bud of the *main* Mandelbrot set that this value is near. Also write the denominator for the bud of the *miniature* Mandelbrot from which this value comes.



Appendix) Practice Exercises

Simplify:

1.1: $(1 - 2\sqrt{-1})(2 + \sqrt{-1})$ **1.2**: $(-2 + 3\sqrt{-1})(1 + \sqrt{-1})$

1.3: $(3 - 4\sqrt{-1})(3 + 4\sqrt{-1})$ **1.4**: $(-2 + \sqrt{-1})(2 + \sqrt{-1})$

2.1: To the nearest hundredth, if $u = 0.6+0.7\sqrt{-1}$, then $u^2 = -0.13+0.84\sqrt{-1}$, $u^3 \approx -0.67+0.41\sqrt{-1}$, $u^4 \approx -0.69-0.22\sqrt{-1}$, $u^5 \approx -0.26-0.61\sqrt{-1}$, and $u^6 \approx 0.27-0.55\sqrt{-1}$. On graph paper, set up a 2-D number plane where each square is 0.2 units wide. The horizontal axis should vary from –1 to 1 while the vertical axis should vary from $-\sqrt{-1}$ to $\sqrt{-1}$. On this number plane, plot and label the points u, u^2, u^3, u^4, u^5, and u^6.

3.1: Let $d = 2+5\sqrt{-1}$. Find $d\sqrt{-1}$. Then find $d\sqrt{-1}\ \sqrt{-1}$. Then find $d\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$. On graph paper make a number plane where each square is 1 unit wide. Make the horizontal axis go from –5 to 5 and the vertical axis go from $-5\sqrt{-1}$ to $5\sqrt{-1}$. Plot and label d, $d\sqrt{-1}$, $d\sqrt{-1}\ \sqrt{-1}$, and $d\sqrt{-1}\ \sqrt{-1}\ \sqrt{-1}$.

Write the rectangular form of:

4.1: 2 cis –120° **4.2**: 4 cis 300° **4.3**: 6 cis 420°

4.4: 3 cis 450° **4.5**: 8 cis –150° **4.6**: 5 cis 270°

Write both the polar and rectangular forms for each product:

5.1: (5 cis 90°)(4 cis 120°) **5.2**: (2 cis 210°)(5 cis 300°)

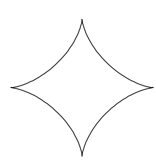
5.3: (cis 180°)(6 cis 240°) **5.4**: (3 cis 300°)(5 cis 240°)

Evaluate:

5.5: $(4 + i)(2 - 3i)$ **5.6**: $(2 + 3i)(2 - 3i)$ **5.7**: $(2 + 3i)(-2 + 3i)$

5.8: $(1 + 5i)(5 - i)$ **5.9**: $(-3 + i)(-3 - i)$ **5.10**: $(1 + i)(-1 + i)$

The equation $z = 3 \text{ cis } \theta + \text{cis } {-3\theta}$ describes the design to the right. Find z for these values of θ:



6.1: 30° **6.2**: 60° **6.3**: 90° **6.4**: 120°

7.1: On graph paper, set up a number plane where the horizontal axis varies from –5 to 5 and the vertical axis varies from $-5i$ to $5i$. Use the head-to-tail method to add 3 and $-4i$. Also use the head-to-tail method to add $-4 - 5i$, $1 + 7i$, and $5 + 3i$. *Label every vector.*

Evaluate:

8.1: $(2 - j)(3i + k)$ **8.2**: $(4i + 2k)(-1 + i)$ **8.3**: $(3i + 4k)(3i + 4k)$

8.4: $|3 - 2i + j - 2k|$ **8.5**: $|-2 + i + j - 5k|$ **8.6**: $|1 + 2i + 2j + k|$

9.1: $(4i)\times(i - j + 2k)$ and $(4i)\bullet(i - j + 2k)$

9.2: $(3i - 2k)\times(2i + j)$ and $(3i - 2k)\bullet(2i + j)$

For each vector, find a unit vector that points the same direction:

10.1: $i - j + 2k$ **10.2**: $-4i + j - 8k$ **10.3**: $3j - 4k$

For each pair of vectors, find the angle between the vectors:

11.1: $3i - 2k$ and $4i + k + j$ **11.2**: $9i + 2j - 6k$ and $-4j + 3k$

For each pair of vectors, find the exact area of the parallelogram formed by them and the approximate angle between them:

12.1: $5k$ and $2i + j - 3k$ **12.2**: $i - 2k$ and $-3i + 4j$

13.1: Rotate $4j - 2k$ about i by 120°

13.2: Rotate $2i + 8j$ about $-k$ by –150°

The Julia set for $z^2 - 0.84 + 0.2i$ is shown at the top of the next page. For the following values of z_0, find z_1 and z_2. Also find $|z_1|$ and $|z_2|$. Round decimals to the nearest hundredth.

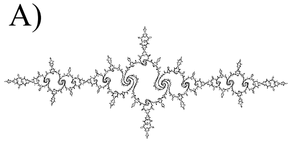
14.1: $z_0 = i$ **14.2**: $z_0 = 1$ **14.3**: $z_0 = -i$ **14.4**: $z_0 = -1$

15.1: For $z^2 + c$, find z_1 and z_2 when $c = 1$ and $z_0 = 1 - i$.

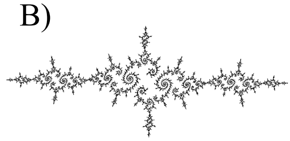
16.1: Shown below is part of the orbit for $z^2 + c$ when $z_0 = 0$ and $c = -1.258 + 0.046i$. Find $|z_{33}|$ to the nearest hundredth. Which Julia set below was produced using this value of c?

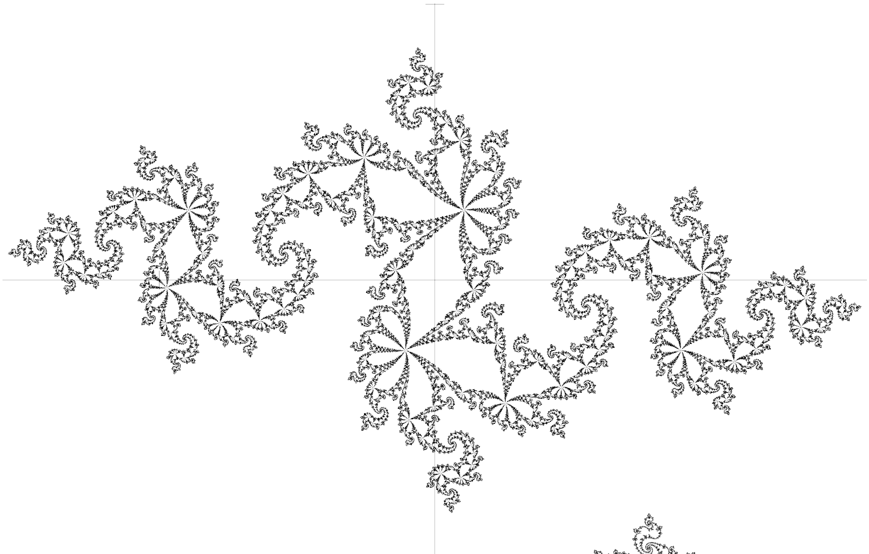
z_{29}	≈	$-1.38 + 0.23i$
z_{30}	≈	$0.60 - 0.58i$
z_{31}	≈	$-1.24 - 0.65i$
z_{32}	≈	$-0.14 + 1.66i$
z_{33}	≈	$-3.99 - 0.41i$

A)

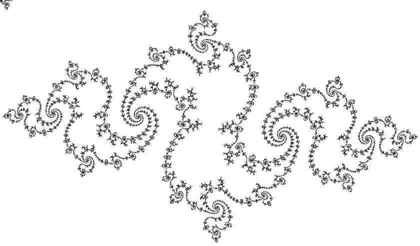


B)

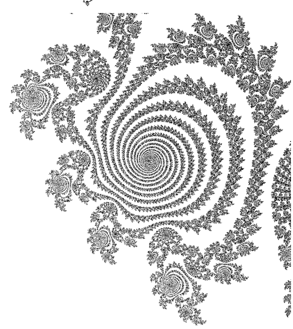




18.1: For the Julia set to the right, *c* = –0.8 + 0.1498*i*. That value is in a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?



19.1: The spiral to the right is near a bud attached to the main cardioid of the Mandelbrot set. What is the denominator of the fraction for that bud?



20.1: For the Julia set on the next page, *c* = –0.75125376315 + 0.02835813853*i*. It is from a bud in a miniature Mandelbrot. To the right is a magnified view of the center of this Julia set. Write the denominator for the bud of the *main* Mandelbrot set that *c* is near. Also write the denominator for the bud of the *miniature* Mandelbrot from which *c* comes. The diagram on the next page is rotated 90°.