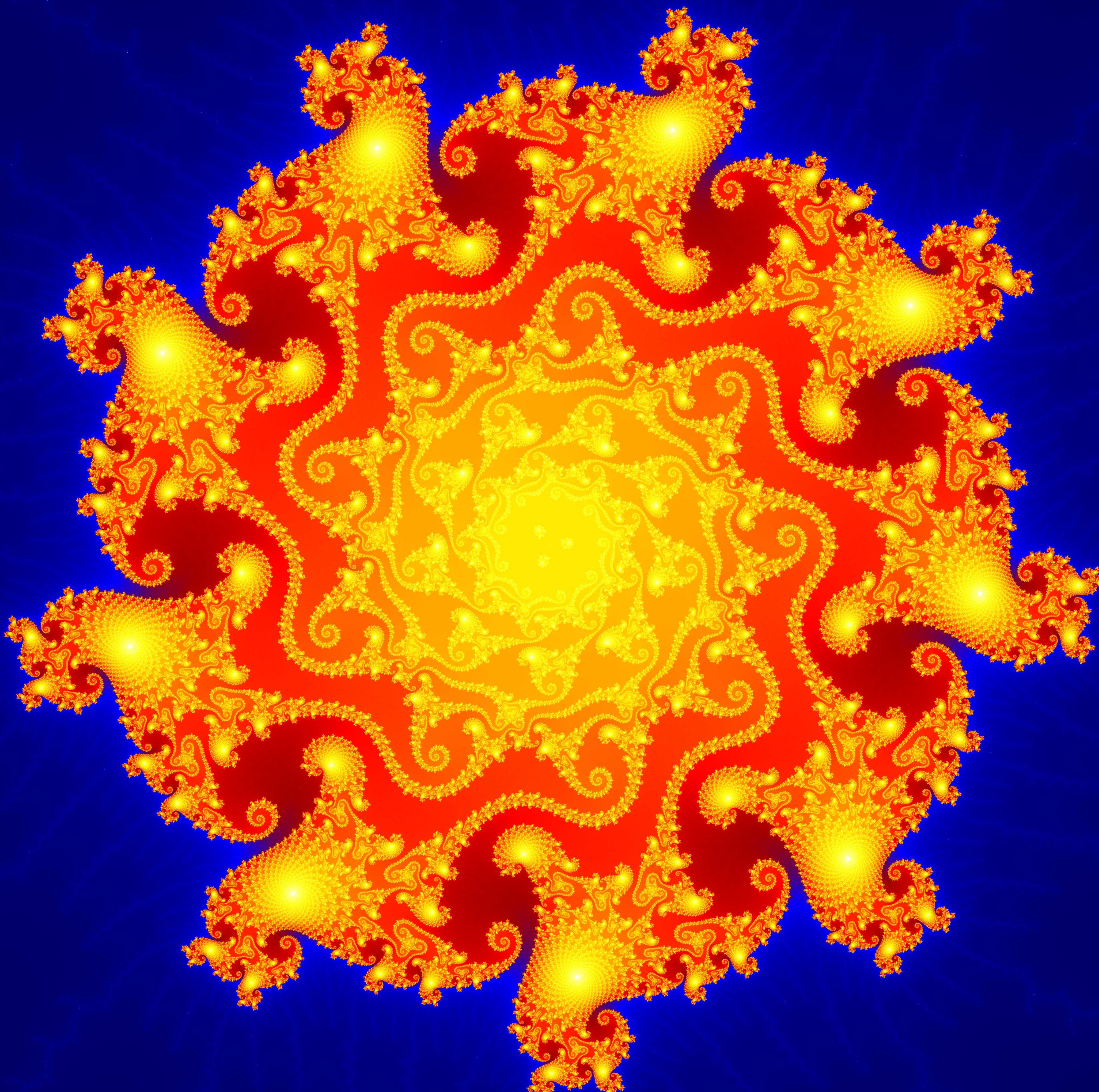
MATHEMATICS



Studying God's Greatness

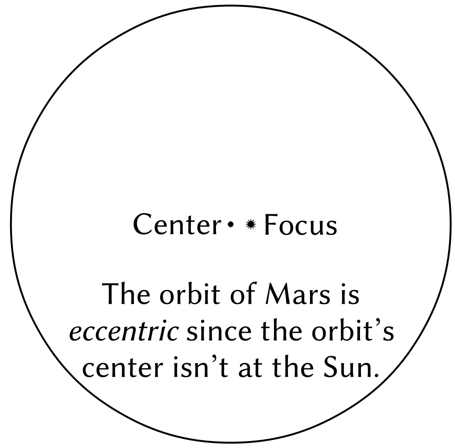
MATHEMATICS

Studying God's Greatness

Johannes Kepler puzzled over the orbit of Mars. Observations showed that Mars did not travel a circular path. What type of path did Mars travel?

Finally, he figured out that a special oval called an *ellipse* agreed well with observations. The Sun is at a special point known as a *focus*. Although the Sun is not at the orbit's center, the Sun is still at a special location.

Kepler enjoyed studying God's creation. Kepler wrote, "I judge it to be piety … to learn myself, and afterwards to teach others too, how great He is in wisdom, how great in power, and of what sort in goodness."[1] A knowledge of mathematics can help us to better appreciate the greatness of the Creator. "The LORD by wisdom hath founded the earth; by understanding hath he established the heavens."[2]



The orbit of Mars looks circular, but it is not a circle. It is extremely close to an ellipse. The Sun is at a focus of the ellipse.

Arithmetic

"He counts the number of the stars; / He calls them all by name. / Great *is* our Lord, and mighty in power; / His understanding *is* infinite."[3] Arithmetic points us to the infinite. When we count, we never come to a largest number. No matter how large of a number we reach, there is always another number after it.

> Ah Lord GOD! behold, thou hast made the heaven and the earth by thy great power and stretched out arm, *and* there is nothing too hard for thee....
>
> Jeremiah 32:17 KJV

When we divide numbers, we might get a decimal number that never ends. If we calculate $1 \div 3$ by hand, it soon becomes clear that the answer is 0.333333333... where there is a never-ending sequence of 3's after the decimal point. We write $0.\overline{3}$ to show that the digit 3 repeats forever.

$$\frac{1}{3} = 0.\overline{3} = 0.33333333333333333333\ldots$$

If $22 \div 7$ is carefully calculated, it becomes clear that the result is $3.\overline{142857}$ which has an infinite number of digits after the decimal point. Also, numbers that seem to stop can be thought of as having an infinite number of digits. The number $1 \div 2$ is 0.500000... where the sequence of 0's continues forever.[4]

$$\frac{22}{7} = 3.\overline{142857} = 3.142857142857142857\ldots$$

$142{,}857 \times 1 = 142{,}857$ $\quad$ $142{,}857 \times 2 = 285{,}714$
$142{,}857 \times 3 = 428{,}571$ $\quad$ $142{,}857 \times 4 = 571{,}428$
$142{,}857 \times 5 = 714{,}285$ $\quad$ $142{,}857 \times 6 = 857{,}142$
$142{,}857 \times 7 = 999{,}999$

$$\begin{array}{rl} & \frac{1}{6} = 0.1666666666\ldots \\ + & \frac{2}{6} = 0.3333333333\ldots \\ \hline & \frac{3}{6} = 0.4999999999\ldots \end{array}$$

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1 Job Kozhamthadam, *The Discovery of Kepler's Laws: The Interaction of Science, Philosophy, and Religion* (Notre Dame, IN: University of Notre Dame Press, 1994), 14.

2 Proverbs 3:19 KJV.

3 Psalm 147:4-5 NKJV.

4 The number $1 \div 2$ can also be written as 0.49999999... as is demonstrated by the addition exercise shown to the right.

Sums

Consider a race between a hare and a tortoise where the hare only goes twice as fast as the tortoise. If the tortoise starts one meter (1 m) in front of the hare, will the hare ever be able to catch up to the tortoise? That question might seem ridiculous, but carefully read the next paragraph.

By the time the hare gets to 1 m, the tortoise will have reached 1.5 m since the tortoise moves half as fast as the hare. Then when the hare reaches 1.5 m, the tortoise will be at 1.75 m. Then when the hare reaches 1.75 m, the tortoise will be at 1.875 m. It appears that whenever the hare reaches where the tortoise had been, the tortoise will have moved a little farther along.

In the first part of the race, the hare travels 1 m. Then it travels 0.5 m. Then it travels 0.25 m, etc. The distance it travels in the first three sections of the race could be written as $(\frac{1}{1}+\frac{1}{2}+\frac{1}{4})$ m.

If the tortoise runs at 0.25 m/s, then the hare plods along at 0.5 m/s. So it takes the hare 2 s to travel 1 m. Then it takes the hare 1 s to travel 0.5 m. Then it takes 0.5 s to travel 0.25 m. So the hare takes $(\frac{2}{1}+\frac{1}{1}+\frac{1}{2})$ s to travel $(\frac{1}{1}+\frac{1}{2}+\frac{1}{4})$ m. How long would it take the hare to go $(\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16})$ m? It would take $(\frac{2}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8})$ s.

To the right is a chart showing the total distance (in meters) and the total time (in seconds) for the hare as it completes each section of the race. The distances form this list: 1, 1.5, 1.75, 1.875, 1.9375, 1.96875, 1.984375, 1.9921875, etc. The times form this list: 2, 3, 3.5, 3.75, 3.875, 3.9375, 3.96875, 3.984375, etc. The distances get closer and closer to 2 m. The times get closer and closer to 4 s.

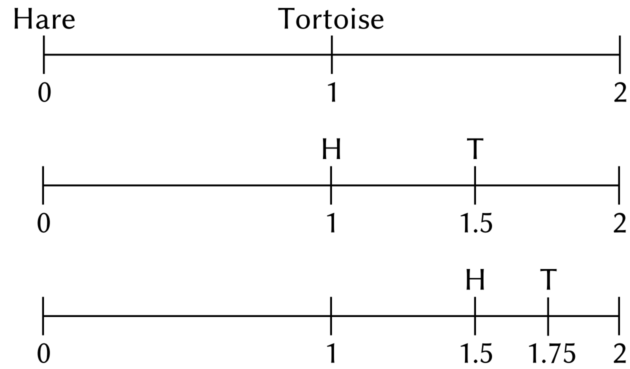
Since the tortoise travels at 0.25 m/s, it goes 1 m in 4 s. Thus at 4 s it will be at 2 m since it started 1 m ahead of the hare. Since the hare moves at 0.5 m/s, it goes 2 m in 4 s. So at 4 s, both the hare and the tortoise are at 2 m. The hare catches up to the tortoise in 4 s.

To the right is another chart showing the total distance and the total time for the hare as it completes each section of the race. The fractions after the minus signs keep getting smaller and smaller as the hare travels more and more sections.

It appears that once the hare reaches 2 m, it will have traveled through an infinity of sections in only 4 s. It goes $(\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots)$ m in $(\frac{2}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\cdots)$ s. It appears that $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$ and $\frac{2}{1}+\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\cdots=4$.

If only the first 100 terms of $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ are summed, the result is less than 2. The same is true if

Fascinating mathematics can help us to better see and appreciate amazing designs in hares and tortoises. This document does not go into those details.



Distances measured in meters

Distance (in meters)		Time (in seconds)	
$\frac{1}{1}$	1	$\frac{2}{1}$	2
$\frac{3}{2}$	1.5	$\frac{3}{1}$	3
$\frac{7}{4}$	1.75	$\frac{7}{2}$	3.5
$\frac{15}{8}$	1.875	$\frac{15}{4}$	3.75
$\frac{31}{16}$	1.9375	$\frac{31}{8}$	3.875

Distance (in meters)		Time (in seconds)	
$\frac{1}{1}$	$2-\frac{1}{1}$	$\frac{2}{1}$	$4-\frac{2}{1}$
$\frac{3}{2}$	$2-\frac{1}{2}$	$\frac{3}{1}$	$4-\frac{1}{1}$
$\frac{7}{4}$	$2-\frac{1}{4}$	$\frac{7}{2}$	$4-\frac{1}{2}$
$\frac{15}{8}$	$2-\frac{1}{8}$	$\frac{15}{4}$	$4-\frac{1}{4}$
$\frac{31}{16}$	$2-\frac{1}{16}$	$\frac{31}{8}$	$4-\frac{1}{8}$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots=2$$

only the first 1,000,000 terms are summed. For the sum to equal 2, *all* the terms must be included in the sum.[5]

Geometry

"O Lord, how great are thy works! *and* thy thoughts are very deep."[6] Galileo wrote that the universe is a book "written in the language of mathematics, and its characters are triangles, circles and other geometric figures…."[7]

Triangles are among the most important geometric figures. Right triangles form an important category of triangles. If a square is drawn on each side of a right triangle, then the sum of the areas of the two smaller squares equals the area of the largest square. That is the Pythagorean theorem.

The diagram to the right shows one way to cut the largest square into five pieces so those pieces can be rearranged to make the two smaller squares. A square the same size as the smallest square is cut out of the center of the largest square. The rest of the largest square is cut into four pieces as shown. These pieces can be rearranged to make the square on the left leg of the right triangle. The numbers show how to rearrange the pieces. For example, piece 5 of the largest square would be positioned against the left leg as shown.

This way of dividing the largest square so its pieces can be rearranged to make the two smaller squares is a type of proof of the Pythagorean theorem. Increasing the rigorousness of the proof would require more work. However, the essential elements of the proof can be ascertained from the diagram.

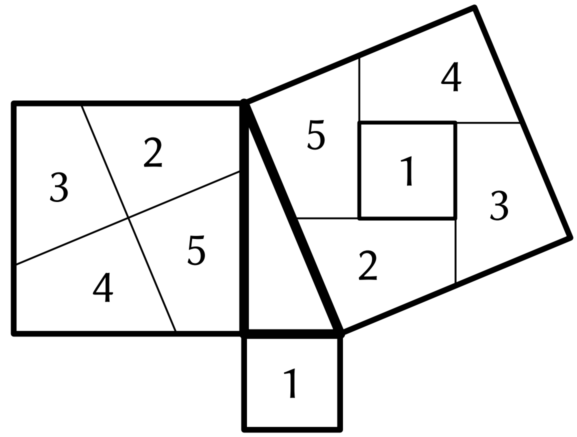
There are numerous ways to prove the Pythagorean theorem, but only one proof is needed for us to know that this concept is truly a theorem. Once we know this powerful concept is true, we can use it with confidence.

Proofs

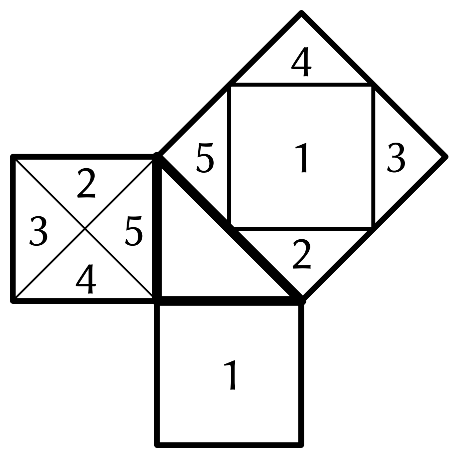
We should remember that mathematical proofs are based on foundational concepts that we accept without proof. A proof might not directly mention all of the foundational concepts upon which its validity rests.

1.9999999999981810105964541435241699218 75

Sum of the first 40 terms



The diagram above shows a demonstration of the Pythagorean theorem for a scalene right triangle. The diagram below shows a demonstration of the Pythagorean theorem for an isosceles right triangle.



$3^2 + 4^2 = 5^2$ $5^2 + 12^2 = 13^2$ $20^2 + 21^2 = 29^2$

$6^2 + 8^2 = 10^2$ $8^2 + 15^2 = 17^2$ $12^2 + 35^2 = 37^2$

If the two legs of a right triangle are 6 units and 8 units, then the hypotenuse is 10 units. The numbers 6, 8, and 10 are a *Pythagorean triple*. Suppose a builder measures along a board 6 ft from a corner and makes a mark. Then he measures 8 ft along another board from the same corner and makes a mark. If the distance between the marks is 10 ft, then the boards approximately form a right angle. Measurement errors limit accuracy.

[5] In the discussion of the race between the hare and the tortoise, infinitely divisible space and time were assumed. Even if that assumption is incorrect, the pure mathematical results still hold. Zeno of Elea (*c.* 490 – *c.* 430 BC) influenced the race story. He mused on a race between Achilles and a tortoise.

[6] Psalm 92:5 KJV.

[7] Galileo, *The Assayer* in *Discoveries and Opinions of Galileo*, trans. Stillman Drake (New York: Anchor Books, 1957), 238.

Proofs of the Pythagorean theorem do not work with right triangles on the surface of a sphere. Not all of the foundational concepts underlying the Pythagorean theorem are valid on a sphere. So the Pythagorean theorem does not work on a sphere. However, there are related concepts which can be used. These concepts can help when dealing with large triangles on the surface of the Earth, even though the Earth is not a perfect sphere.

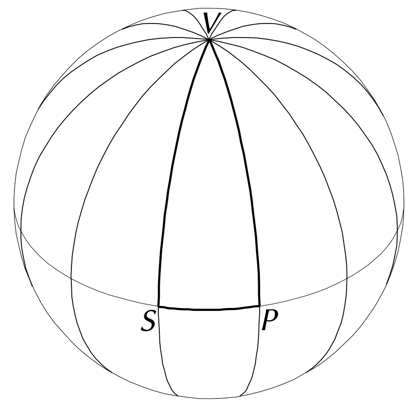
One foundational concept assumed in each proof is that the method of logic used in the proof is valid. In discussing the foundations of mathematics, James R. Newman wrote, "And finally, what of logical reasoning itself? Who is to vouch for it?"[8] Those who do not accept truth from God severely limit their ability to understand physical and spiritual realities. "The fear of the LORD *is* the beginning of knowledge...."[9]

Albert Einstein claimed that "one should expect the world to be chaotic, not to be grasped by thought in any way."[10] He also claimed, "The most incomprehensible thing about the world is that it is comprehensible."[11] These claims came from his inaccurate view of the world. He accepted the existance of order[12] in the universe, and he apparently realized that this order did not logically follow from his view of the world.

Logical Thinking

Why does logical thinking work? "In the beginning was the Word, and the Word was with God, and the Word was God. The same was in the beginning with God. All things were made by him; and without him was not any thing made that was made."[13] God made man in His image and gave man dominion.[14]

A physicist wrote that "certainly it is hard to believe that our reasoning power was brought, by Darwin's process of natural selection, to the perfection



On this sphere, adjacent lines of longitude are separated by 30°. Triangle *PVS* is isosceles. The angles at *S* and *P* are both right angles. The measure of angle *V* is 30°. Do you see how to draw a triangle on this sphere so that all of the angles in the triangle are right angles?

> The heavens declare the glory of God; and the firmament sheweth his handywork. Day unto day uttereth speech, and night unto night sheweth knowledge.
> Psalm 19:1-2 KJV

> By the word of the LORD were the heavens made; and all the host of them by the breath of his mouth.
> Psalm 33:6 KJV

> For great *is* the LORD, and greatly to be praised: he also *is* to be feared above all gods. For all the gods of the people *are* idols: but the LORD made the heavens.
> I Chronicles 16:25-26 KJV

> But the natural man receiveth not the things of the Spirit of God: for they are foolishness unto him....
> I Corinthians 2:14a KJV

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8 James R. Newman, ed., *The World of Mathematics* (New York: Simon and Schuster, 1956), 3:1935.

9 Proverbs 1:7a KJV.

10 James Nickel, *Mathematics: Is God Silent?* (Vallecito, CA: Ross House Books, 1990), 69.

11 John Archibald Wheeler, "Albert Einstein" in *The World Treasury of Physics, Astronomy, and Mathematics*, ed. Timothy Ferris (Boston: Little, Brown and Company, 1991), 564.

12 On September 7, 1944, he wrote to a friend, "You believe in the God who plays dice, and I in complete law and order in a world which objectively exists...." (Albert Einstein, "From *Letters to Max Born*" in *World Treasury*, 809).

13 John 1:1-3 KJV. The Greek for *Word* is *logos* (related to *logic*).

14 Genesis 1:26-28. Since then, man sinned. Not only are we limited because of being made a little lower than the angels, we are also limited by clouded thinking that has resulted from sin.

which it seems to possess."[15] He claimed that "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and … there is no rational explanation for it."[16]

If we accept what God has revealed, we do have a good explanation for the enormous usefulness of mathematics. "The fear of the LORD *is* the beginning of wisdom: and the knowledge of the holy *is* understanding."[17] Wisdom proclaims, "The LORD possessed me in the beginning of his way, before his works of old."[18] Wisdom declares, "When he prepared the heavens, I *was* there.…"[19] Johannes Kepler wrote that God "has fashioned us in his own image, that we may come into participation of the same reasoning with himself."[20]

Ellipse

The diagram to the right shows a string attached to two points, each of which is a *focus* (plural: *foci*). A pencil is held in such a way so as to keep the string taut. If the pencil is carefully moved while the string is kept taut, the pencil traces out an *ellipse*. This can be done by placing paper on a flat sheet of wood or cardboard. Then at each focus, a tack is pressed through the string and paper into the wood or cardboard.

If the string is kept the same length, but the foci are moved closer together, the ellipse becomes more like a circle. As the foci are moved closer to the center, the *eccentricity* of the ellipse decreases. If the foci were placed on top of each other, the resulting design would be a circle instead of an ellipse.

The eccentricity of an ellipse is the ratio of the center-focus distance to the center-vertex distance. For the orbit of Mars, the foci are so close to the center that the ellipse is nearly circular. The eccentricity of that orbit is around 0.0934. A circle's eccentricity is 0.

[15] Eugene P. Wigner, "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" in *World Treasury*, 528.

[16] Wigner, 527. "It is, as Schrödinger (1933) has remarked, a miracle that in spite of the baffling complexity of the world, certain regularities in the events could be discovered." (529). Wigner's puzzlement may have come in part from his belief in the *invention* of mathematical concepts and rules (528). We should be impressed with the greatness of our Creator as we consider our ability to *discover* and *name* (cf. Genesis 2:19-20) and *appreciate* beautiful patterns from Him.

[17] Proverbs 9:10 KJV.

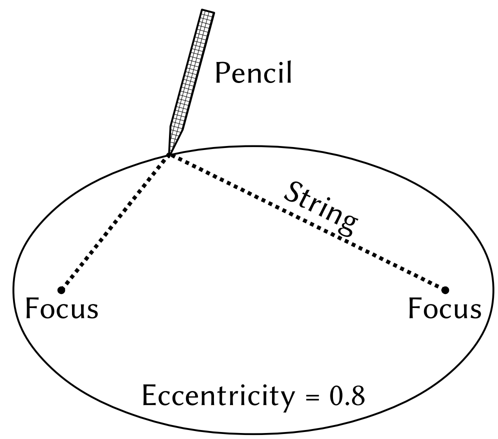
[18] Proverbs 8:22 KJV.

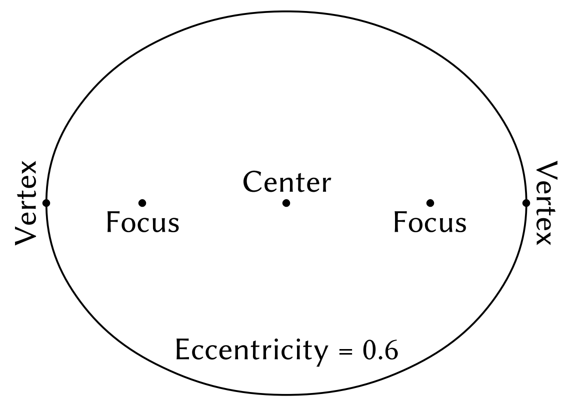
[19] Proverbs 8:27a KJV.

[20] Johannes Kepler, "Kepler's Correspondence in 1599," trans. Louisa H. Richardson, *The Sidereal Messenger* 6(4) (Apr. 1887), 137.

So God created man in his *own* image, in the image of God created he him; male and female created he them.
Genesis 1:27 KJV

Through faith we understand that the worlds were framed by the word of God, so that things which are seen were not made of things which do appear.
Hebrews 11:3 KJV

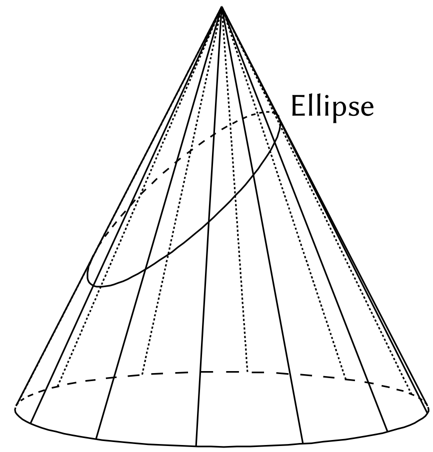




He hath made the earth by his power, he hath established the world by his wisdom, and hath stretched out the heaven by his understanding.
Jeremiah 51:15 KJV

I will praise thee; for I am fearfully *and* wonderfully made: marvellous *are* thy works; and *that* my soul knoweth right well.
Psalm 139:14 KJV

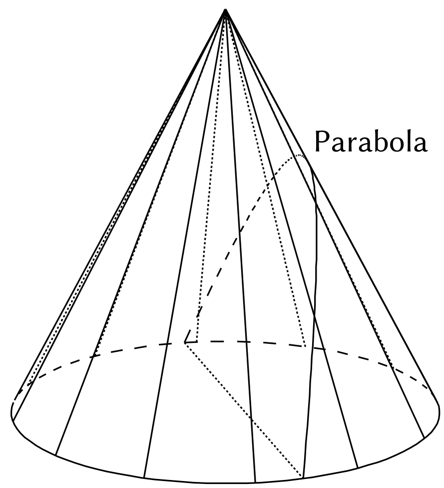
The eccentricity of Earth's orbit is around 0.0167, much smaller than the eccentricity of the orbit of Mars. The center of Earth's orbit is around 2,500,000 km from the focus.[21] That distance is relatively small for an astronomical distance. We can be thankful for the small eccentricity of Earth's orbit. This small eccentricity means that the distance from the Earth to the Sun varies relatively little.



Cone

To the right is a diagram of a portion of a cone. The base of the cone is a circle even though it looks elliptical from this perspective. The cone has been sliced at a slant, and the curve at the location of the slice is an ellipse. From this perspective, the ellipse looks more eccentric than it is.

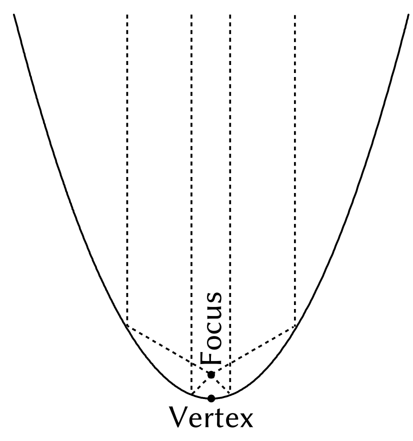
If the cone is cut at steeper angles, the ellipse becomes more eccentric. If the cone is cut so that the slice is parallel to a side of the cone, the curve is no longer an ellipse. Instead, the curve is a *parabola*. Only part of a parabola is shown in the cone to the right. The cone actually continues forever, and the parabola also continues forever.[22]



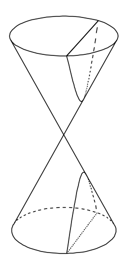
The parabola shown in the cone has its vertex at the top. The parabola shown below the cone is turned so its vertex is at the bottom. A parabola has only one focus and only one vertex.

Reflections

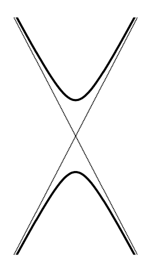
Parabolas are quite useful. A flashlight may have a reflector that is parabolic. Imagine a silvery surface formed by spinning a parabola about the line that goes through its focus and vertex. If a light were at the focus, then the reflected light rays would be parallel to each other. To the right are lines that show how light rays from the focus would reflect from the parabola to form parallel rays of light.



This property of parabolas can be used to collect incoming light. Many telescopes have a mirror with a parabolic shape. Light coming from a distant object is practically parallel. So if a parabolic mirror is aimed toward incoming light, then light reflected from the mirror will be reflected toward the focus.



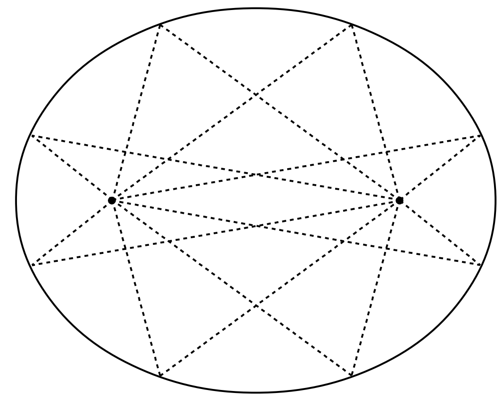
To the right is a hyperbola with two lines that are its *asymptotes*. The farther a portion of the hyperbola is from the center of the hyperbola, the closer that portion is to an asymptote.



[21] Around 1,550,000 miles.

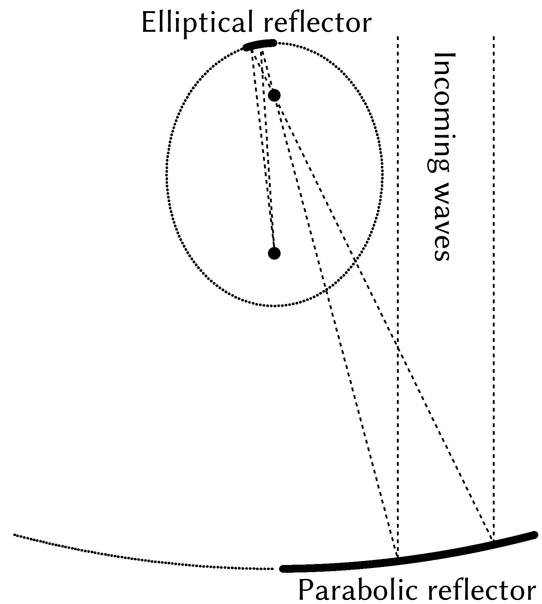
[22] If the cone is cut at a steeper angle, the result is a *hyperbola*. The cone continues infinitely both up and down. The top and bottom portions, *nappes*, are joined at a single point, the *vertex*, as shown to the right, Both the top and bottom nappes are sliced. So a hyperbola has two parts called *branches*.

An ellipse also has interesting reflective properties. The diagram to the right shows the reflection of light rays within a reflective elliptical surface. Light rays that leave one focus reflect off the elliptical surface and go to the other focus. Sound waves reflect similarly as light waves. In an elliptical room, a person standing at one focus may be able to whisper and be clearly heard at the other focus.



The reflective properties of ellipses and parabolas can be combined in the construction of telescopes. Consider the elliptical and parabolic reflectors shown to the right. *The top focus of the ellipse is also the focus of the parabola*. The dotted curves are not a part of the structure. These dotted curves help to show the ellipse and parabola behind this type of telescope design.

Two examples of incoming radio waves are shown to the far right. These rays bounce off a reflector with a parabolic shape. They travel through the top focus and bounce off a reflector with an elliptical shape. Then they go toward the bottom focus of the ellipse to be detected. A massive radio telescope[23] was made with this type of design so that detectors and support structures could be placed outside of the field of vision of the telescope. Doing this decreased the possibility for interference in the detection of signals.

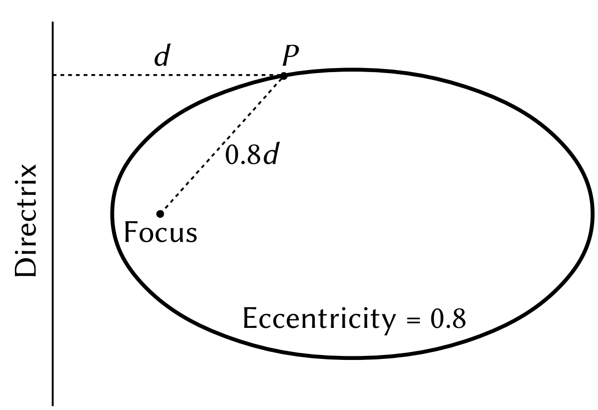


Conic Sections

Curves that can be formed by slicing a cone are called *conic sections*. Conic sections help us to better understand the greatness of the Creator. These curves help us to understand patterns in the motions of planets and asteroids and comets. Conic sections are also used to help design telescopes that help us to see deeply into space and get a better understanding of the wonders that God has made. "The works of the LORD *are* great, sought out of all them that have pleasure therein."[24]

Greeks studied conic sections many centuries before the time of Kepler. Then when Kepler puzzled over the orbit of Mars, it finally occurred to him to use an ellipse to describe the path of Mars. The ellipse to the right is much more eccentric than the orbit of any planet in the solar system. However, there are comets that travel orbits even more eccentric than this ellipse.

For any point P on the ellipse below, the distance from P to the focus is 0.8 of the distance from P to a special line known as the *directrix*. The diagram shows only one focus and one directrix. There is another directrix associated with the other focus of the ellipse.



Kepler discovered a way to describe how fast Mars goes around the Sun. The rule that he discovered for Mars works for any planet, asteroid, or comet.

[23] Near Green Bank, WV.
[24] Psalm 111:2 KJV.

Three sections in the ellipse to the right have been labeled with numerals. *Each section has the same area*. This ellipse represents the orbit of a comet. The comet travels most slowly when it is farthest from the focus. So when it travels from *A* to *B*, it moves a little faster than when it is farthest from the focus. Then as it travels from *C* to *D*, it moves even faster. It moves the fastest at point *E* which is the point on the ellipse closest to the focus.

The time it takes the comet to go from *A* to *B* is the same as the time it takes it to go from *C* to *D* or the time it takes it to go from *E* to *F*. The line segment from the focus to the comet sweeps out *equal areas in equal times*. When the comet is closer to the focus, the line segment is shorter. However, the faster speed of the comet compensates for the shorter line segment so that equal areas are swept out in equal amounts of time.

Kepler discovered amazing patterns in the motion of Mars. These patterns not only apply to Mars, they also apply to other planets orbiting the Sun, as well as to comets and asteroids. These patterns also apply to satellites orbiting the Earth in elliptical paths. For such orbits, the Earth is at a focus. The Creator designed the universe in such a way that there are patterns that humans can discover.[25]

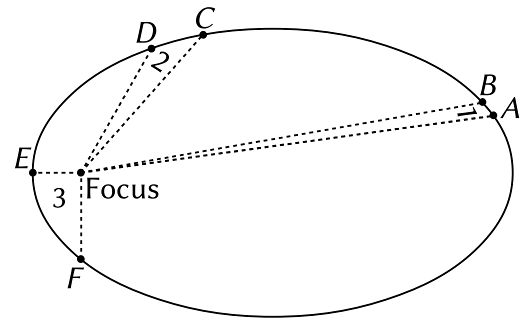
L-Systems

To the right is a mathematical fern. The Creator has made many fascinating designs, and ferns display an important design feature called *self-similarity*. The mathematical fern to the right is made of smaller parts that are approximately similar to the whole fern. One of those parts is outlined with a rectangle.

The fern and snowflake designs shown to the right were made using *L-systems*[26] which are named after

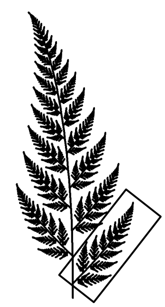
[25] Wigner wrote, "The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve." (540). We can understand Who gave this gift.

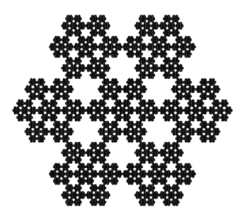
[26] Fern L-system:
```
FernM {~ ;order 50
  Axiom \90A
  D=@1.11D@I1.11
  A=D[W]D\1B
  B=D[!W]D\1A
  W=/45:8&A
}
```
Flake L-system:
```
Flake6Fill { ;order 5
  Angle 6
  Axiom F+KF+KF+KF+KF+KF+K
  K=@I3F+KF+KF+[|K]KF+KF+KF+K@3
  F=G
}
```



Kepler discovered the area law *before* he knew that the shape of the orbit is an ellipse. However, the area law is called Kepler's *second* law. The ellipse law is called his *first* law, even though he discovered it *after* the area law. Kepler's laws are not exact. They are really good approximations.

Lord, thou *art* God, which hast made heaven, and earth, and the sea, and all that in them is....
Acts 4:24b KJV





Every good gift and every perfect gift is from above, and cometh down from the Father of lights....
James 1:17a KJV

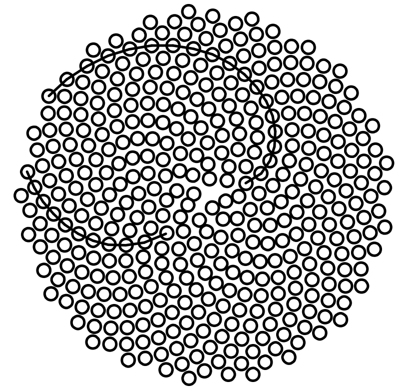
For the Lord giveth wisdom: out of his mouth *cometh* knowledge and understanding.
Proverbs 2:6 KJV

L-system designs in this document were drawn with *Arcnel*.

Aristid Lindenmayer (1925-1989) who introduced the concept in 1968. He helped to write *The Algorithmic Beauty of Plants* which was first published in 1990.[27]

If an object is "rough" and has some level of self-similarity, it is a *fractal*.[28] Both the mathematical fern and the snowflake design are fractals. To the right is a design that is not a fractal. Designs like this appear in sunflowers and other plants. This design was created using an L-system.[29]

Two curves on this design mark two types of spirals found in the pattern of the miniature circles. Careful counting shows 21 spirals of the one type and 34 spirals of the other type. The numbers 21 and 34 are in the Fibonacci sequence, an important sequence[30] of numbers related to designs in God's creation. "O LORD, how manifold are thy works! in wisdom hast thou made them all: the earth is full of thy riches."[31]



Fibonacci Sequence

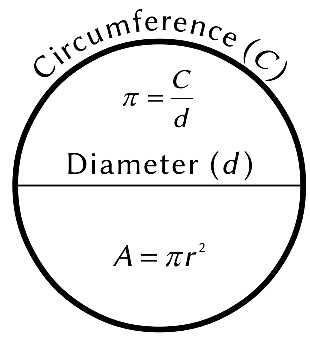
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

3 + 5 = 8, 5 + 8 = 13, 8 + 13 = 21, etc.

Algebra

Algebra simplifies the statements of many concepts in arithmetic and geometry. The symbolism of algebra helps us understand mathematical concepts. Algebraic symbolism makes it easier to discover new concepts.

The area of a circle equals the product of the square of its radius and the ratio of the circumference of a circle to its diameter. Contrast the previous sentence with this simple formula: $A = \pi r^2$. The sentence has 26 words with a total of 105 letters. The algebraic formula has only 5 symbols. Algebra simplifies life.



π = 3.14159265358979323846264338327950...

The decimal for π never ends and never repeats. No obvious pattern has been found in the decimal for π. However, there are ways to express the value of π so that there is a pattern. To the right are some infinite expressions for π that contain patterns.

$$\pi = 2\cdot\tfrac{2}{1}\cdot\tfrac{2}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{5}\cdot\tfrac{6}{5}\cdot\tfrac{6}{7}\cdot\tfrac{8}{7}\cdot\tfrac{8}{9}\cdot\tfrac{10}{9}\cdot\tfrac{10}{11}\cdots$$

$$\pi = 2\left(1+\tfrac{1}{3}+\tfrac{1\cdot 2}{3\cdot 5}+\tfrac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}+\tfrac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}+\cdots\right)$$

$$\pi = 3\cdot\tfrac{2}{\sqrt{2+\sqrt{3}}}\cdot\tfrac{2}{\sqrt{2+\sqrt{2+\sqrt{3}}}}\cdot\tfrac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}\cdots$$

$$\pi = 4\left(\tfrac{1}{1}-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\cdots\right)$$

$$\pi = \sqrt{6\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)}$$

$$\pi = \sqrt[3]{32\left(\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots\right)}$$

$$\pi = \cfrac{4}{1+\cfrac{1^2}{3+\cfrac{2^2}{5+\cfrac{3^2}{7+\cfrac{4^2}{9+\cfrac{5^2}{11+\cdots}}}}}}$$

The conciseness of algebraic expressions helps us to focus on the concepts involved. Suppose a circle with a certain area needs to be made. How could we

[27] The other writer was Przemyslaw Prusinkiewicz. The book contains numerous designs created using L-systems.

[28] In 1975, Benoit B. Mandelbrot (1924-2010) coined *fractal*.

[29]
```
Florets21L34 {~ ;order 400; also try order 1114
   Axiom [K][B]/137.507764:1@V
   K=K\137.507764['@Q1M@I@@.7(360]
   B=:19&A
   A=B\7.663044['@Q20M@I@@.7\90M".]
   V=:32&U
   U=V/4.736024['@Q8M@I@@.7\90M".]
}
```

[30] As the sequence advances, dividing a Fibonacci number by the previous such number results in a value that converges to the golden ratio, 1.6180339887498948482045868... = $(1 + \sqrt{5}) \div 2$.

[31] Psalm 104:24 KJV.

calculate the radius of the circle? Using algebra, it can be shown that $r = \sqrt{A \div \pi}$. Someone proficient at algebra can discover that formula with very little mental effort by algebraically manipulating the formula $A = \pi r^2$.

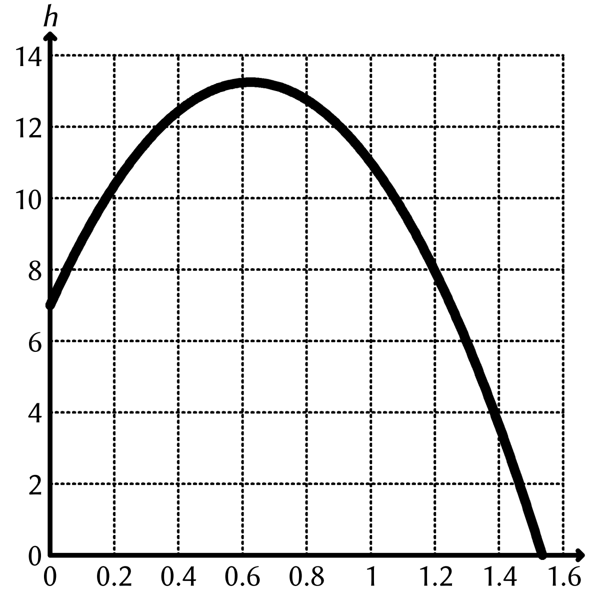
To whom then will you liken God? …
It is He who sits above the circle of the earth,
And its inhabitants *are* like grasshoppers,
Who stretches out the heavens like a curtain,
And spreads them out like a tent to dwell in.
Isaiah 40:18a, 22 NKJV

Expressions

Many patterns in God's creation can be described using algebraic expressions. Suppose a softball is thrown straight up at a speed of 20 ft/s from a height of 7 ft above the ground. If the height (h) is measured in feet and the time (t) in seconds, then the height is approximated by the expression $h = -16t^2 + 20t + 7$.

In the equation $h = -16t^2 + 20t + 7$, the *variables* are t and h. If $t = 1$ s, the equation shows that $h = 11$ ft. It can help to consider many different values for the variables in an algebraic expression and to make a graph of the result.

To the right is a graph for $h = -16t^2 + 20t + 7$. It can be mathematically proven that the curve is a portion of a parabola, one of the conic sections studied by Greeks over 2000 years ago. The labels show that the horizontal axis is the t-axis and the vertical axis is the h-axis.



$h = -16t^2 + 20t + 7$

By studying the graph, we can see that when $t = 1$ s the ball is going down instead of up. We can also see that the maximum height of the ball is a little more than 13 ft and occurs shortly after 0.6 s. It also appears that the ball hits the ground sometime after 1.5 s.

Using calculus, it is easy to show that the ball's maximum height occurs at 0.625 s. At that time, the ball's height is 13.25 ft.

Can we do better than just estimate the time the ball hits the ground? We can. When the ball hits the ground, $h = 0$ ft. If we put that value into the algebraic expression, we get $0 = -16t^2 + 20t + 7$. How can we find a value for t that will make the right side of the equation equal 0?

Solutions

Once students have advanced far enough in the study of algebra, they learn how to solve equations like $0 = -16t^2 + 20t + 7$. To the right is an expression that can be used to find a value for t. Evaluating that expression results in two values for t. The one value is close to –0.285 s, and the other value is approximately 1.535 s.

$$0 = at^2 + bt + c$$
$$\Downarrow$$
$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-20 \pm \sqrt{20^2 - 4(-16)(7)}}{2(-16)}$$

In this situation, the value that makes the most sense is 1.535 s. That value is an approximation. The equation $0 = -16t^2 + 20t + 7$ has the exact solution shown to the right. Of course, we should remember that $h = -16t^2 + 20t + 7$ is actually an approximate relation

$$t = \frac{5 + \sqrt{53}}{8} \text{ s}$$

between h and t. In that expression, the *coefficient* of t^2 is –16. That number is not exact. Its value ranges from around –16.13 at the Earth's poles to around –16.04 at the equator. Also, the value gets closer to 0 as the altitude increases. We should also remember that the equation $h = -16t^2 + 20t + 7$ ignores air resistance.

Gravity

Algebra provides an efficient method for stating many patterns our Creator has designed.[32] To the right is an algebraic equation that describes the force (F) of gravity between two masses, m_1 and m_2. If the two masses are uniform solid spheres, the value of d is the distance between the centers of the spheres. In words, the force of gravity is directly proportional to the product of the masses and inversely proportional to the square of the distance between the masses.

In the formula for gravitational force, the *constant of proportionality* is G. If distance is measured in meters (m), mass is measured in kilograms (kg), and force is measured in newtons (N), then G is around 0.0000000000667 N·m²/kg². The small value of G is a result of the weakness of the gravitational force. For gravity to be significant, at least one of the masses must be extremely massive.

Isaac Newton discovered the law of gravity. The work of Johannes Kepler helped prepare the way for Newton's discovery. One pattern Kepler discovered can be stated this way: the square of the time (T) taken by a planet to orbit the Sun is directly proportional to the cube of its average distance (d) from the Sun.

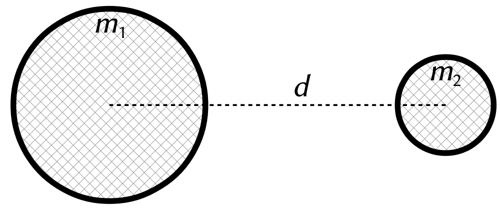
The constant G from the law of gravity can be used when writing Kepler's discovery algebraically as shown to the right. In that equation, M is the mass of the Sun.

That equation can also be used if the mass being orbited is not the Sun.[33] For example, Jupiter has four

The relation $h = -16t^2 + 20t + 7$ may be written as $h(t) = -16t^2 + 20t + 7$. We read $h(t)$ as "h of t." This notation treats the relation as describing a *function* called h. Using this type of notation, we can write $h(0) = 7$. This tells us that when $t = 0$, the value of the function is 7. Also, $h(1) = 11$ shows that $t = 1$ results in a value of 11.

Newton's Law of Gravity

$$F = \frac{Gm_1m_2}{d^2}$$



Kepler's First Law
A planet's orbit is an *ellipse* with the Sun at a *focus.*

Kepler's Second Law
The line segment from a planet to the Sun sweeps out *equal areas in equal times.*

Kepler's Third Law
The *square* of the *time* it takes a planet to orbit the Sun *is directly proportional to* the *cube* of its average *distance* from the Sun.

$$T^2 = \frac{4\pi^2}{GM}d^3$$

Circles approximately describe the paths of the planets. Ellipses do better. The law of gravity does better yet. The general theory of relativity does even better. Layers of patterns help us to discover order in what God has made.

It is often more useful to use the law of gravity than the general theory of relativity. At times, ellipses are more useful than the law of gravity. Layers of patterns provide a robust set of tools for studying God's amazing creation.

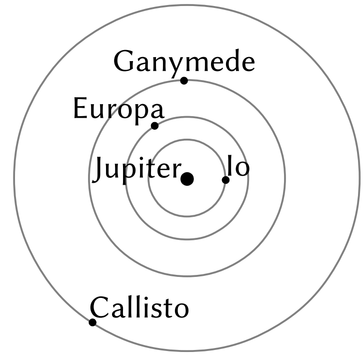
$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)}d^3$$

[32] Wigner wrote that "it is not at all natural that 'laws of nature' exist, much less that man is able to discover them." (531). He wrote of "the succession of layers of 'laws of nature'" (531). These patterns point to a supernatural Creator. Our discovery abilities do as well. Wigner noted that "the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena." (534).

[33] If the mass being orbited is not significantly more massive than the mass that is orbiting, then the modified equation shown to the right can be used for greater accuracy. In this equation, m_1 and m_2 are the masses of the objects which are orbiting their common center of mass. See p. 418 in *Physics for Scientists and Engineers*, 4th ed. (Philadelphia: Saunders College Publishing, 1996) by Raymond A. Serway.

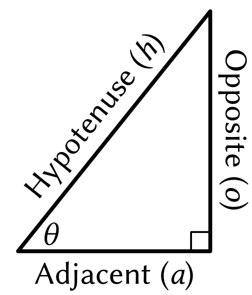
moons that are easily visible with a telescope. For those moons, M is the mass of Jupiter. By observing the motions of those moons, the mass of Jupiter can be calculated with the help of that equation. We can use algebra to find the formula for M shown to the right.

$$M = \frac{4\pi^2 d^3}{GT^2}$$



Right Triangles

Triangles are extremely useful. Algebra helps us to work with triangles. For an acute angle in a right triangle, we will consider three important ratios. The ***sine*** of the angle is the ratio of the length of the leg *opposite* the angle to the length of the *hypotenuse*. The ***cosine*** of the angle is the ratio of the length of the leg *adjacent* to the angle to the length of the *hypotenuse*. The ***tangent*** of the angle is the ratio of the length of the leg *opposite* the angle to the length of the leg *adjacent* to the angle.



If the measure of the acute angle is symbolized by the Greek letter θ (theta), then these three ratios can be written as $\sin\theta$, $\cos\theta$, and $\tan\theta$. The expression "$\sin\theta$" means "sine of theta." These three ratios are examples of *trigonometric functions*.

$$\sin\theta = \frac{o}{h} \qquad \cos\theta = \frac{a}{h} \qquad \tan\theta = \frac{o}{a}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

With a calculator we can find **sin *θ*** if we know ***θ***. At times, we may know the ratio **sin *θ*** and want to find the angle ***θ***. At those times, an *inverse* trigonometric function is useful. The inverse sine function is written as $\sin^{-1}$. So if r is the ratio of a right triangle's leg to the length of the hypotenuse, then $\sin^{-1} r$ is the measure of the angle opposite that leg.

$$\theta = \sin^{-1}\frac{o}{h} \qquad \theta = \cos^{-1}\frac{a}{h} \qquad \theta = \tan^{-1}\frac{o}{a}$$

The superscript −1 is *not* an exponent.
It indicates an *inverse* function.

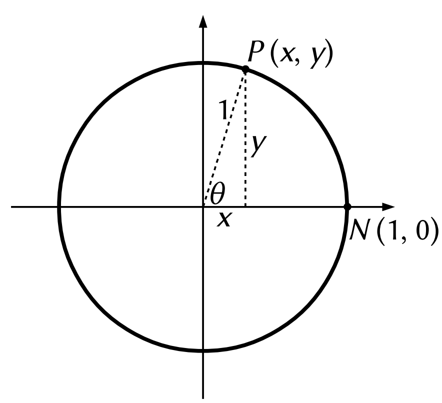
Sometimes it is useful to discuss $(\sin\theta)^2$. That concept is written as $\sin^2\theta$. Similarly, $\cos^2\theta$ and $\tan^2\theta$ stand for the squares of $\cos\theta$ and $\tan\theta$.

For the right triangle shown, the Pythagorean theorem shows that $a^2 + o^2 = h^2$. Through the use of algebra, it can be shown that $\cos^2\theta + \sin^2\theta = 1$.

$$a^2 + o^2 = h^2$$

$$\cos^2\theta + \sin^2\theta = 1$$

It may appear that trigonometric functions would only work with the acute angles found in right triangles. However, the definitions of $\sin\theta$, $\cos\theta$, and $\tan\theta$ can be extended so these functions work with any angle.[34] This increases the usefulness of the functions.



$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x}$$

[34] To the right is a *unit circle*. This circle is centered at the origin and has a radius of 1 unit. Point P on the circle has coordinates (x, y). The angle to P is θ. By studying the triangle, we can see that $\cos\theta = x$, $\sin\theta = y$, and $\tan\theta = y/x$. We can extend the definitions of trigonometric functions so those three equations work for any point P on the unit circle.

The ratio of the length of arc NP to the radius of the circle is a unitless number that is the value of θ in *radians*. To the right are expressions for $\sin\theta$ and $\cos\theta$ when θ is in radians and where $2!$ ("2 factorial") is $2\cdot1$, $3! = 3\cdot2\cdot1$, $4! = 4\cdot3\cdot2\cdot1$, etc.

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$0! = 1$
$1! = 1\cdot0! = 1$
$2! = 2\cdot1! = 2$
$3! = 3\cdot2! = 6$
$4! = 4\cdot3! = 24$

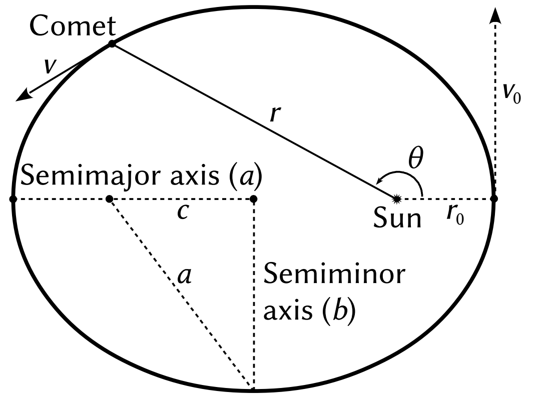
Orbits

To the right is the orbit of a comet. The Sun is at a focus of the ellipse. When the comet is closest to the Sun, its distance from that focus is r_0 and its speed is v_0, which is represented by an arrow pointing the direction the comet is traveling at that point.

The current distance of the comet from that focus is r, and its current speed is v. The angle through which it has traveled to reach its current location is θ. As the comet goes farther from the Sun, r increases and v decreases. To the right are formulas for calculating r and v when θ is known.[35] These formulas work for planets and asteroids as well as comets.

The long axis of the ellipse is its *major axis*. A *semimajor axis* is half of the major axis. The length of a semimajor axis is a. The short axis is the *minor axis*. Half of that axis is a *semiminor axis* whose length is b. The distance from the center to a focus is c. Do you see from the diagram that $r_0 = a - c$? The farthest the comet gets from the Sun is $r_{180} = a + c = 2a - r_0 = r_0 + 2c$. The average distance of the comet from the Sun is a, half of the length of the major axis.

It can be proven that the distance from a focus to an end of the minor axis is a. The Pythagorean theorem shows that $a^2 = b^2 + c^2$. If two variables in that equation are known, algebra can be used to find the other one. The Creator designed the universe and us in a way that allows us to partly understand great things He has done.



$$\text{Eccentricity} = e = \frac{c}{a}$$

$$r = \frac{(1+e)r_0}{1+e\cos\theta} \qquad e = \frac{r_0 v_0^2}{GM} - 1$$

$$v = \frac{v_0}{1+e}\sqrt{1+2e\cos\theta+e^2}$$

$$r_0 = a - c$$

$$r_{180} = a + c = 2a - r_0 = r_0 + 2c$$

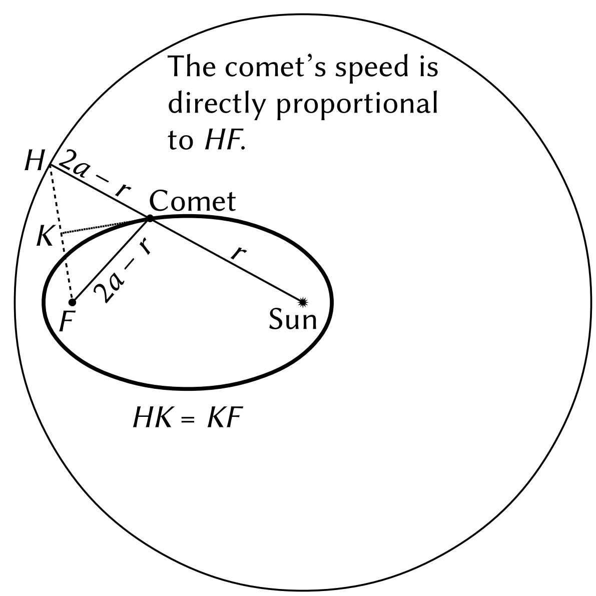
$$v_{180} = \frac{1-e}{1+e}v_0 = \frac{a-c}{a+c}v_0 = \frac{r_0}{r_{180}}v_0$$

$$a^2 = b^2 + c^2$$

Speed

One way to visualize the speed of a comet is to use a diagram like the one shown to the right.[36] The large circle is centered at the Sun and has a radius equal to the length of the major axis of the elliptical orbit. So the radius of the large circle is $2a$.

The distance from the Sun to the comet is r, and the distance from the Sun to H is $2a$. The ellipse's foci are



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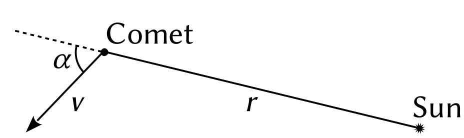
[35] The r and e formulas shown beside each other are on pp. 895-896 in *Calculus and Analytic Geometry*, 9th ed. (Reading, MA: Addison-Wesley Publishing Company, 1996) by George B. Thomas, Jr. and Ross L. Finney. A form of the v formula is on p. 765 in *Calculus with Analytic Geometry*, 5th ed. (New York: John Wiley & Sons, Inc., 1995) by Howard Anton.

If a comet's position and velocity are known where α (alpha) is the indicated angle in the diagram to the right, then e and r_0 can normally be found with the formulas below the diagram. If $e = 1$, then the path is a parabola, $\theta = 180° - 2\alpha$, and r_0 can be found using $r_0 = r\sin^2\alpha$ instead of the formula to the far right which can't be used when $e = 1$ since $0 \div 0$ would occur.

[36] Keith Kendig, *Conics* (Mathematical Association of America, 2005), 339-340.

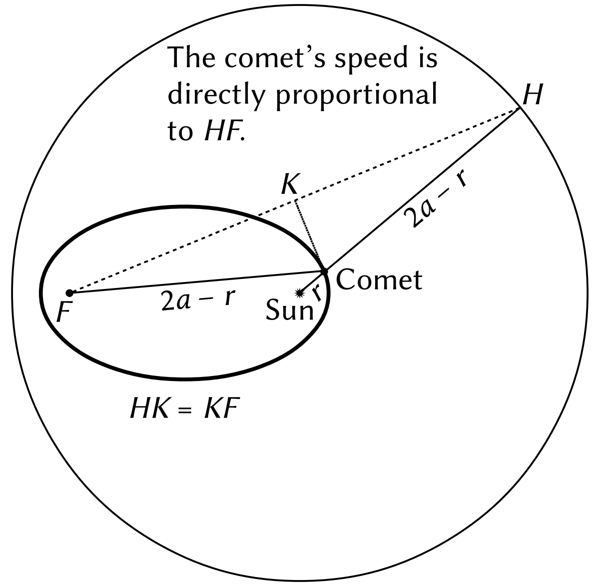


$$e = \sqrt{1 - \frac{rv^2(2GM - rv^2)\sin^2\alpha}{(GM)^2}} \qquad r_0 = \frac{r(1-e)GM}{2GM - rv^2}$$

at F and the Sun. Point K is the midpoint of line segment HF. The line segment from the comet to K is tangent to the ellipse. So this line segment can be used to determine the direction of travel since the direction the comet moves is tangent to its elliptical orbit.

The comet's speed is directly proportional to HF. As the comet goes away from the Sun, line segment HF shrinks and the comet slows down. As the comet goes toward the Sun, HF increases and the comet speeds up.

It is fascinating that there are two ways to visualize the speed of a comet or planet or asteroid. One way is to use this type of diagram. Another way is to visualize equal areas being swept out in equal times. Kepler has been quoted as declaring, "The chief aim of all investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics."[37]



For *God* gives wisdom and knowledge and joy to a man who *is* good in His sight....
Ecclesiastes 2:26a NKJV

Two-Dimensional Numbers

The study of algebra led to the discovery of special numbers that have enhanced our understanding of God's creation. Consider the equation $x^3 - 6x + 4 = 0$. This is a special type of *cubic equation*. If a cubic equation has the form $x^3 + px + q = 0$, then a value for x can be found using the formula shown to the right. In this case, $p = -6$ and $q = 4$.

$$x^3 + px + q = 0$$
$$\Downarrow$$
$$x = \sqrt[3]{-\tfrac{q}{2} + \sqrt{\left(\tfrac{q}{2}\right)^2 + \left(\tfrac{p}{3}\right)^3}} + \sqrt[3]{-\tfrac{q}{2} - \sqrt{\left(\tfrac{q}{2}\right)^2 + \left(\tfrac{p}{3}\right)^3}}$$

Using this formula to solve $x^3 - 6x + 4 = 0$ gives the solution shown to the right. That solution contains the expression $\sqrt{-4}$. What does $\sqrt{-4}$ mean? No number on the number line can be squared to get –4. Although $\sqrt{-4}$ is not on the number line, it is a number. So it must be somewhere other than the number line.

$$x = \sqrt[3]{-2 + \sqrt{-4}} + \sqrt[3]{-2 - \sqrt{-4}}$$

Algebra shows us that $\sqrt{-4}$ can be written as $2\sqrt{-1}$. So the solution shown earlier can be rewritten as shown to the right. That solution contains $\sqrt{-1}$ which cannot be found on the number line.

$$x = \sqrt[3]{-2 + 2\sqrt{-1}} + \sqrt[3]{-2 - 2\sqrt{-1}}$$

Now consider $(1 + \sqrt{-1})(1 + \sqrt{-1})(1 + \sqrt{-1})$. If we carefully multiply that product, we get $-2 + 2\sqrt{-1}$. So $(1 + \sqrt{-1})^3 = -2 + 2\sqrt{-1}$. Similar calculations show that $(1 - \sqrt{-1})^3 = -2 - 2\sqrt{-1}$.

Using these results, the solution shown earlier can be rewritten and then simplified as shown to the right. These calculations show that $x = 2$. We can substitute 2 for x in $x^3 - 6x + 4 = 0$ to see that 2 is a solution. So

$$x = \sqrt[3]{\left(1 + \sqrt{-1}\right)^3} + \sqrt[3]{\left(1 - \sqrt{-1}\right)^3}$$
$$x = 1 + \sqrt{-1} + 1 - \sqrt{-1}$$
$$x = 2$$

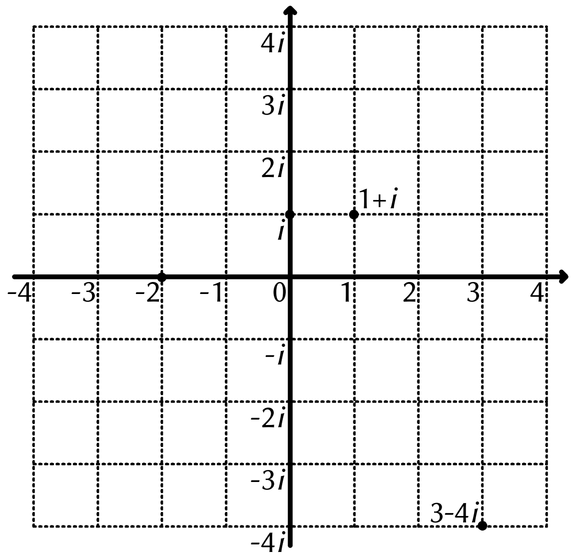
We ought to be impressed by our Creator's greatness as we study beautiful patterns He has enabled us to see.

[37] Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1990), 1:231. Not only does mathematics help us to see beautiful patterns in the physical world, mathematics also contains beautiful patterns.

although $\sqrt{-1}$ is not on the number line, it can be used to find a solution that is on the number line.

Where is $\sqrt{-1}$ if it isn't on the number line? It is on the *number plane*. To the right is a diagram of the number plane. In that diagram, the letter i is used to symbolize $\sqrt{-1}$. Thus $i^2 = -1$.

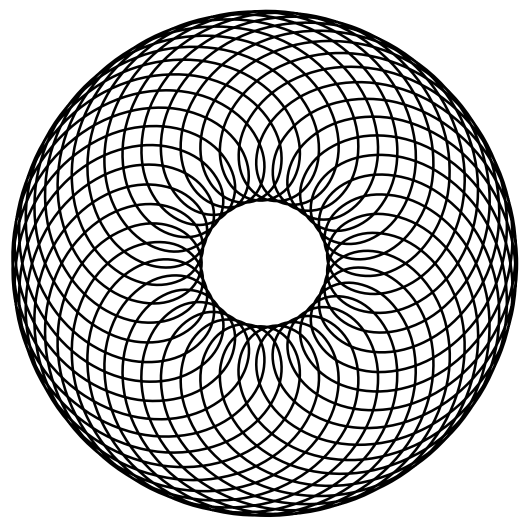
The horizontal axis of the number plane is the number line. The numbers -2, i, $1 + i$, and $3 - 4i$ are plotted on this number plane. The number line is one-dimensional. The number plane is two-dimensional. Two-dimensional numbers help us to enhance our understanding of God's creation.



Simplicity

The use of two-dimensional numbers can simplify descriptions of various curves. When working with such numbers, the expression **cos θ + i sin θ** is often useful. This expression is abbreviated as **cis θ**.

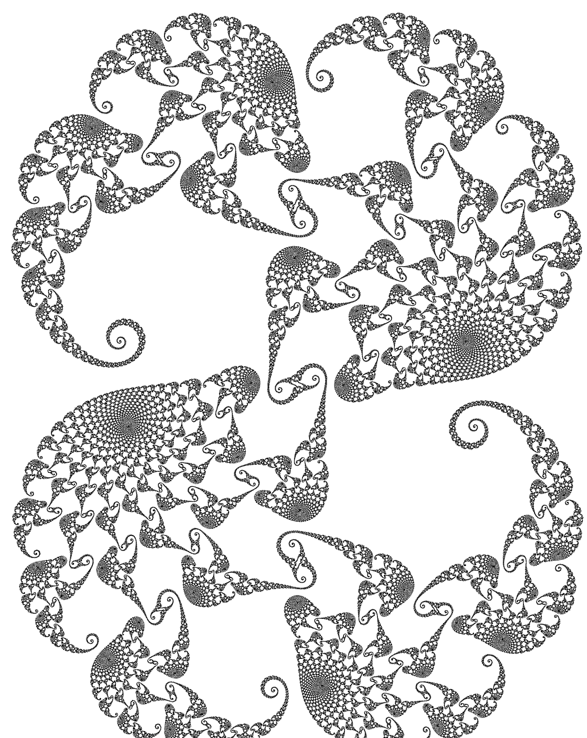
The expression 2.5cis 3.9θ + 1.5cis 15θ describes the design to the right.[38] If values of θ are substituted into 2.5cis 3.9θ + 1.5cis 15θ, then each value of θ produces a number which can be plotted on the number plane. For example, if $\theta = 10°$, then the expression becomes 2.5cis 39° + 1.5cis 150° which is close to $0.64 + 2.32i$. The design to the right can be made by plotting many values of θ ranging from 0° to 1200°.



The number plane is not shown with the design. To see how this design is positioned in the number plane, imagine placing the design on the number plane shown above it. The center of the design is at 0, and the right side[39] is at 4. The top[40] is near $0.052 + 3.999i$ which is close to $4i$. The bottom[41] of the design is approximately $0.052 - 3.999i$ which is close to $-4i$. The curve reaches farthest to the left at two locations.[42] The locations are around $-3.996 - 0.104i$ and $-3.996 + 0.104i$. Each of those locations is near -4.

Sets

To the right is a design that comes from analyzing the simple expression $z^2 + c$ when $c = 0.265 + 0.003i$.



[38] This design can be drawn by a pen on the end of a rotating rod that is attached to another rotating rod. The one rod is 2.5 units long and rotates 3.9 times a second. The other rod is 1.5 units long and rotates 15 times a second.

The long curve with the title on page 1 can be drawn using three rods. The curve's formula: cis θ + cis $-\theta$ + 0.025cis 11θ.

[39] This occurs when $\theta = 0°$.

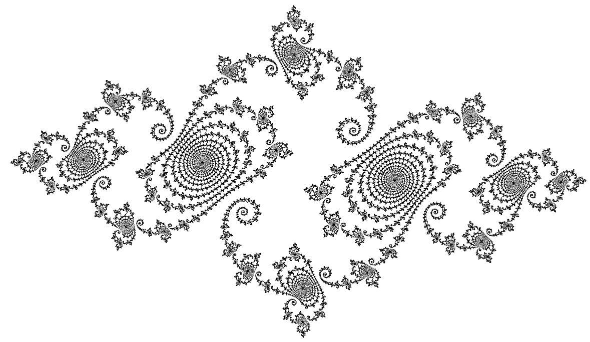
[40] θ is slightly less than 1038.05°.

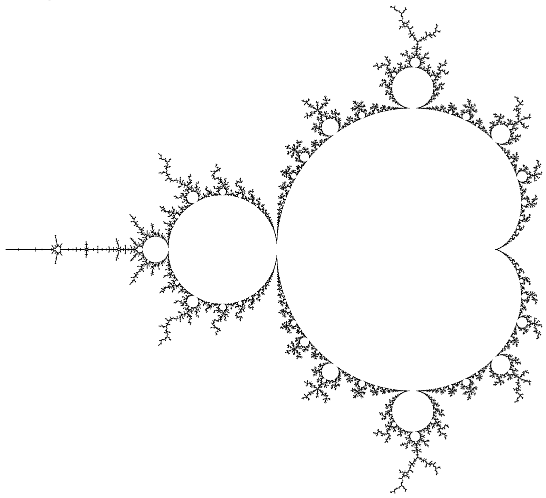
[41] θ is slightly more than 161.95°.

[42] θ is near 323.91° or 876.09°.

That design is a diagram of a *Julia set*, named after the French mathematician Gaston Julia. Different values of c in $z^2 + c$ produce different Julia sets.

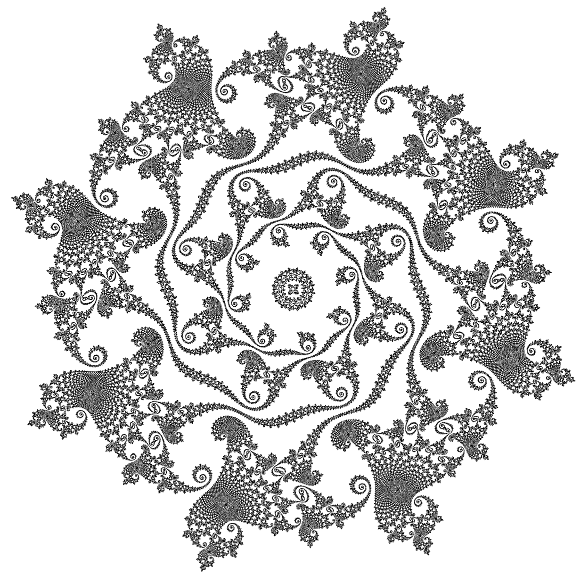
To the right is another Julia set for $z^2 + c$. In this case, $c = -0.75 + 0.09i$. These Julia sets[43] are fractals, as is the design below.





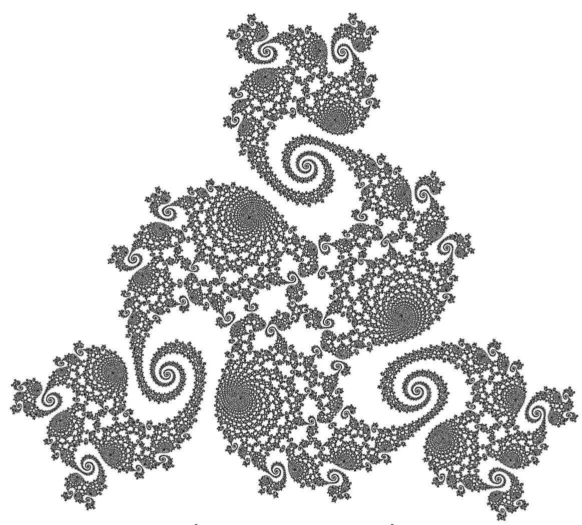
The Julia set for $z^2 - 0.77 - 0.06i$ is next to the page number at the bottom of the page. Also, that Julia set and a mirror image are at the top of page 1 on either side of the title.

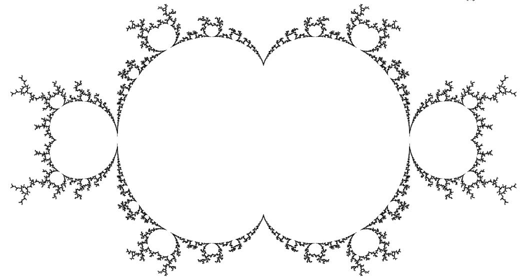
The design shown above is an outline of the *Mandelbrot set* for $z^2 + c$. To the right is a small portion[44] of the border of the Mandelbrot set. The design shown does not include all of the Mandelbrot set that exists in this region of the number plane. Many thin filaments are not shown. This includes thin filaments that connect various parts of this design, as well as filaments that connect this design with the rest of the Mandelbrot set.



The Mandelbrot set is a special map used to discover Julia sets with various characteristics. It is an amazing journey to magnify a small part of the border of the Mandelbrot set, and then to magnify a tiny part of what has been magnified, and then to magnify a tiny part of what has been magnified, etc. Theoretically, the border of the set could be magnified forever.

Not only are there Julia sets for $z^2 + c$, there are also Julia sets for $z^3 + c$. To the right is the Julia set for $z^3 + 0.0785 + 0.7842i$. Below that is an outline of the Mandelbrot set for $z^3 + c$ (the outline has been rotated 90°). Not only do two-dimensional numbers help us to analyze designs in the universe, these numbers are also connected to fascinating designs that motivate us to think about infinity.





[43] Julia set designs in this document show *approximate* locations of points in Julia sets, not mathematically exact Julia sets. The same concept holds for Mandelbrot set designs.

[44] The height of the design is around 1.19×10^{-37}. The center is around –1.94153627464209273403366716128049953769885 – 0.00006416624461654898081175393933334427486*i*.

Vectors

God created us to live in three spatial dimensions. People searched for three-dimensional numbers and did not find them. Then William R. Hamilton discovered four-dimensional numbers on October 16, 1843. These numbers consist of a one-dimensional part joined with a three-dimensional part.

Hamilton[45] called these numbers *quaternions*. To help understand them, consider the three arrows shown to the right. The arrows i, j, and k are each 1 unit long. The arrows i and j are in the plane of this page. The arrow k points out of the page. All three arrows are perpendicular to each other even though they might not look so when drawn on this two-dimensional page.

The three arrows represent numbers that obey the following rule: $i^2 = j^2 = k^2 = ijk = -1$. Algebra can be used on this rule to discover that $ij = k$ and $ji = -k$.[46] Did you notice that ij does not equal ji? When working with quaternions, the order of multiplication is important!

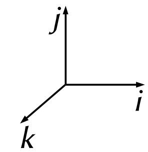
The number $q = 3 + 5i + j - k$ is an example of a quaternion. The one-dimensional part is 3 which can be plotted on a number line. The three-dimensional part is $5i + j - k$ which can be viewed as an arrow in space.

The three-dimensional part of a quaternion is a *vector*. Vector math is a powerful tool used in studying the world God has made. Also, people use vectors to help design objects for our three-dimensional world.[47]

Calculus

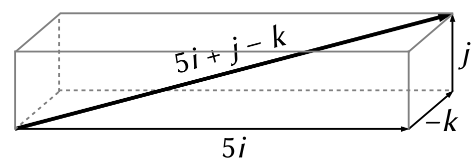
To the right is a graph of the height (h) of a softball. This ball was thrown upward from a height of 7 ft and at a speed of 20 ft/s. The equation for the curve is $h = -16t^2 + 20t + 7$ where time (t) is measured in seconds. The line was drawn approximately tangent to the curve at $t = 0.8$ s.

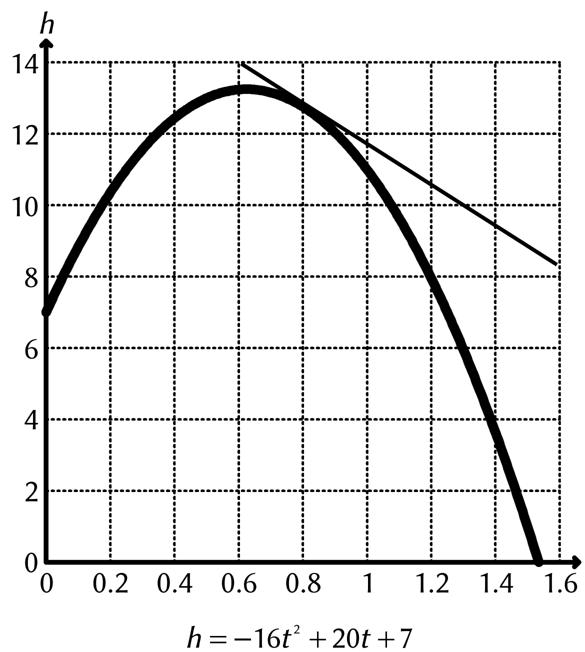
People have used intriguing words when referring to numbers that were unfamiliar to them. Years ago, a negative number was called *absurd*. Even today, a square root of a negative number may be called *imaginary*.



$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k \qquad jk = i \qquad ki = j$$
$$ji = -k \qquad kj = -i \qquad ik = -j$$





$$h = -16t^2 + 20t + 7$$

$$\left(\cos\tfrac{\theta}{2} + u\sin\tfrac{\theta}{2}\right)v\left(\cos\tfrac{\theta}{2} - u\sin\tfrac{\theta}{2}\right)$$

$$\left(\cos\theta + u\sin\theta\right)v$$

---

[45] "At the age of five Hamilton could read Latin, Greek, and Hebrew. At eight he added Italian and French; at ten he could read Arabic and Sanskrit and at fourteen, Persian." (Kline, 2:777). "He was deeply religious and this interest was most important to him." (Kline, 2:778).

[46] Start with $ijk = -1$ and multiply on the right by k on each side to get $ijkk = -1k$. This is $ij(-1) = -1k$ since $k^2 = -1$. Multiply both sides by -1 to get $ij = k$.

Now start with $-1 = ijk$. Multiply on the left by i on each side to get $i(-1) = iijk$. This is $i(-1) = -1jk$ since $i^2 = -1$. Multiply both sides by -1 to get $i = jk$. Multiply on the left by j on each side to get $ji = jjk$. Since $j^2 = -1$, this simplifies to $ji = -k$.

[47] To rotate vector v by an angle of θ about vector u where u is a unit vector (a vector exactly 1 unit long), we can do the first quaternion product shown to the right. If we know that v is perpendicular to u, we may do the second quaternion product.

The slope of the line was estimated to be –5.7 ft/s. For this graph, the slope of a tangent line at a certain time is the velocity of the ball at that time. So a slope of –5.7 ft/s means that the ball is falling at 5.7 ft/s.

Calculus can be used to find the exact slope of a line tangent to this curve. The point at which the line touches the curve is the point of tangency. The slope of the line is the slope of the curve at that point. Since the slope is the velocity of the ball, v will be used to represent the slope. Calculus shows that $v = -32t + 20$. If $t = 0.8$ s, this equation shows that $v = -5.6$ ft/s. So the estimate was fairly close.

Differentiation

The calculus process that comes up with the equation for the slope of a curve is *differentiation*. If $h = -16t^2 + 20t + 7$ is differentiated with respect to t, the result is $v = -32t + 20$. The expression $-32t + 20$ is the *derivative* of h with respect to t and may be symbolized by dh/dt or h'. We read dh/dt as "*dh dt*" and h' as "*h* prime."

In the Algebra section, it was shown that the ball hits the ground when t is approximately 1.535 s. Substituting that time into $v = -32t + 20$ gives a slope of –29.12 ft/s.

Integration

Part of the parabola $y = -x^2 + 4x$ is shown to the right. What is the area between the x-axis and this part of the parabola? By carefully studying the graph, it can be estimated that the area is between 10 and 11 square units.

Using calculus, the exact area can be calculated. Calculus shows that the exact area is $10.\overline{6}$ square units. The calculus process that finds areas is *integration*. To the right is an equation that contains an *integral* for calculating the area between the x-axis and the parabola. According to this equation, $10.\overline{6}$ is the value of the integral.

The processes of integration and differentiation are inverse processes. When integrating an expression, it may help to find an *antiderivative* of the expression. An antiderivative is an expression whose derivative is the expression to be integrated. If an antiderivative of an expression can be found, then that antiderivative can be used to integrate the expression.[48]

The letter e is used for the eccentricity of ellipses and other curves that can be formed by slicing cones. The letter e also symbolizes a special number used in calculus and other areas of mathematics. The decimal for e is endless: 2.71828182845904523536028747135... An exact expression for e is shown below.

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

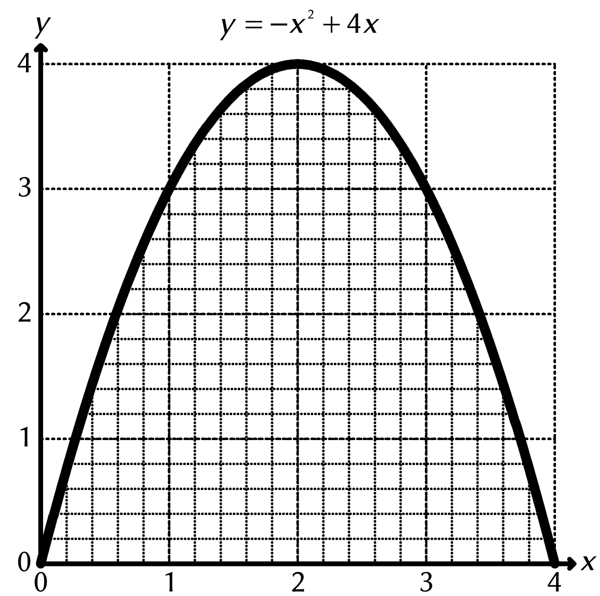
The equation below is simple and profound. It relates five important numbers, and it uses addition, multiplication, and exponentiation.

$$e^{i\pi} + 1 = 0$$

$$h = -16t^2 + 20t - 7$$

$$\Downarrow$$

$$\frac{dh}{dt} = -32t + 20$$



$$\int_0^4 (-x^2 + 4x)\,dx = 10.\overline{6}$$

The equation below relates differentiation and integration. In that equation, the function f' is the derivative of the function f. Thus f is an antiderivative of f'.

$$\int_a^b f'(t)\,dt = f(b) - f(a)$$

$$g(x) = -\tfrac{1}{3}x^3 + 2x^2 \quad \Rightarrow \quad g'(x) = -x^2 + 4x$$

[48] For $-x^2 + 4x$, an antiderivative is the $g(x)$ shown to the right. So the integral of $-x^2 + 4x$ from 0 to 4 is $g(4) - g(0) = 10.\overline{6}$.

Mathematical Order

Using calculus, Newton's law of gravity can be used to help derive Kepler's three laws of planetary motion. To the right is Kepler's second law written using calculus notation.[49] The Creator designed the universe and us in such a way that we can discover order from Him that is in the universe. There is order visible to anyone willing to look. However, there are patterns that do not become visible to us until we understand certain mathematical concepts.

James Clerk Maxwell[50] studied electrical and magnetic concepts, and he discovered *electromagnetic waves*. He calculated the speed of these waves by mathematically manipulating physics concepts. *The speed that he calculated was close to the measured speed for light*. So he concluded that light is an electromagnetic wave.

Maxwell developed a system of equations that can be used to study electromagnetic concepts. To the right are four equations[51] that show one way of writing Maxwell's mathematical concepts. The bold symbols represent vector concepts. These equations provide a way to describe a portion of the work of the Creator. "For by him were all things created, that are in heaven, and that are in earth, visible and invisible, whether *they be* thrones, or dominions, or principalities, or powers: all things were created by him, and for him: And he is before all things, and by him all things consist."[52]

$$\frac{dA}{dt} = \frac{1}{2} r_0 v_0$$

The left side of the equation above refers to the instantaneous rate of change of area with respect to time. The value on the right side is constant for an ideal elliptical orbit. So the rate of change of area is constant. Thus equal areas are swept out in equal intervals of time.

And God said, Let there be light: and there was light.
Genesis 1:3 KJV

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$$

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

The speed of light (c) is $1 \div \sqrt{\mu_0 \varepsilon_0}$. The equation $E = mc^2$ relates mass (m) with the energy (E) equivalent to that mass. In the special theory of relativity, c is a constant for all frames of reference for which the law of inertia holds.

Conclusion

There is much more mathematics that we can use to study God's greatness. For example, Boolean algebra is foundational to the design of modern digital electronic circuits, such as those found on computer chips. Digital electronic circuits are often used to gather and process details about what our Creator has made.

Algebra teaches symbolic thinking skills. These skills have many applications. Both computer code and DNA contain symbols. Also, algebra can help to solve certain puzzles that may arise when writing software.

Numbers point us to the concept of infinity. The numbers 1, 2, 3, 4, … are *cardinal numbers*. The numbers 1st, 2nd, 3rd, 4th, … are *ordinal numbers*.

Let us consider the sizes of mathematical sets. The techniques used may seem elementary when studying

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Above is a *truth table*. Truth tables are useful in analyzing concepts in Boolean algebra. The letters p and q represent statements that are either true or false. The symbols $\wedge$, $\vee$, and ~ stand for the concepts *and*, *or*, and *not*. The symbol $\vee$ represents an *inclusive or* instead of an *exclusive or*. The last column of this truth table shows an *exclusive or*.

Buy the truth, and sell *it* not….
Proverbs 23:23a KJV

… and the truth shall make you free.
John 8:32b KJV

[49] Thomas and Finney, 896.
[50] He lived 1831-1879. Pronounce *Clerk* as *Clark*.
[51] Serway, 925, 996. The equation $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is on p. 1000.
[52] Colossians 1:16-17 KJV.

small sets. However, these techniques are quite useful when considering the sizes (cardinalities) of larger sets.

The *cardinality* of the set $K = \{2, 7, 8\}$ is 3. We can write $card(K) = 3$. Consider the set $L = \{1, 2, 3\}$. The diagram to the right shows a *mapping* from L onto K where *each* element of K is an *image* of at least one element of L. This type of mapping is an *onto mapping*. The existence of a mapping from L onto K shows that $card(L) \geq card(K)$.

$$\begin{array}{rccc} L\!: & 1, & 2, & 3 \\ & \downarrow & \downarrow & \downarrow \\ K\!: & 2, & 7, & 8 \end{array}$$

$card(L) \geq card(K)$

"$\geq$" means "is greater than or equal to"

There is also a mapping from K onto L as shown to the right. This means that $card(K) \geq card(L)$. Since $card(K) \geq card(L)$ and $card(L) \geq card(K)$, it can be concluded that $card(L) = card(K)$.

$$\begin{array}{rccc} K\!: & 2, & 7, & 8 \\ & \downarrow & \downarrow & \downarrow \\ L\!: & 1, & 2, & 3 \end{array}$$

$card(K) \geq card(L)$

Consider the set $M = \{1, 2, 3, 4, 5\}$. There is a mapping from M onto K. Therefore $card(M) \geq card(K)$. It is impossible for there to be a mapping from K onto M. So it is not true that $card(K) \geq card(M)$. Therefore $card(K) < card(M)$.

$$\begin{array}{rccccc} M\!: & 1, & 2, & 3, & 4, & 5 \\ & \searrow & \downarrow & \downarrow & \downarrow & \swarrow \\ K\!: & & 2, & 7, & 8 & \end{array}$$

$card(M) \geq card(K)$

The sets K, L, and M are small enough that we can quickly determine their cardinalities by counting. We easily see that $card(K) = card(L) = 3 < card(M) = 5$.

Consider the *natural numbers*, $\mathbb{N} = \{1, 2, 3, \dots\}$, and the *positive even numbers*, $\mathbb{E}^+ = \{2, 4, 6, \dots\}$. There is a mapping from $\mathbb{N}$ onto $\mathbb{E}^+$. If e is in $\mathbb{E}^+$, then $e \div 2$ in $\mathbb{N}$ can be mapped onto e. So $card(\mathbb{N}) \geq card(\mathbb{E}^+)$.

$$\begin{array}{rcccc} \mathbb{N}\!: & 1, & 2, & 3, & \dots \\ & \downarrow & \downarrow & \downarrow & \\ \mathbb{E}^+\!: & 2, & 4, & 6, & \dots \end{array}$$

$card(\mathbb{N}) \geq card(\mathbb{E}^+)$

Also, there is a mapping from $\mathbb{E}^+$ onto $\mathbb{N}$. If n is in $\mathbb{N}$, then $2n$ in $\mathbb{E}^+$ can be mapped onto n. This shows that $card(\mathbb{E}^+) \geq card(\mathbb{N})$. So $card(\mathbb{E}^+) = card(\mathbb{N})$ since it is also true that $card(\mathbb{N}) \geq card(\mathbb{E}^+)$ as shown above. This shows that the number of even numbers equals the number of natural numbers. The cardinal number that equals $card(\mathbb{N})$ is written as $\aleph_0$ (read as "aleph null"). Aleph ($\aleph$) is the first letter of the Hebrew alphabet.

$$\begin{array}{rcccc} \mathbb{E}^+\!: & 2, & 4, & 6, & \dots \\ & \downarrow & \downarrow & \downarrow & \\ \mathbb{N}\!: & 1, & 2, & 3, & \dots \end{array}$$

$card(\mathbb{E}^+) \geq card(\mathbb{N})$

The set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of *integers*. Clearly, $card(\mathbb{Z}) \geq card(\mathbb{N})$ since every element of $\mathbb{N}$ is in $\mathbb{Z}$. Also, $card(\mathbb{N}) \geq card(\mathbb{Z})$ since there is a mapping from $\mathbb{N}$ onto $\mathbb{Z}$ as is shown to the right. Thus $card(\mathbb{Z}) = card(\mathbb{N}) = \aleph_0$. So the number of integers is the same as the number of natural numbers.

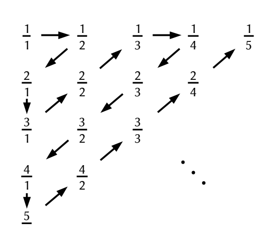
$$\begin{array}{rcccccccc} \mathbb{N}\!: & 1, & 2, & 3, & 4, & 5, & 6, & 7 & \dots \\ & \swarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \searrow & \\ \mathbb{Z}\!: & 0, & -1, & 1, & -2, & 2, & -3, & 3, & \dots \end{array}$$

$card(\mathbb{N}) \geq card(\mathbb{Z})$

If z is an integer, then there is a natural number mapped onto it by the following rules: if $z < 0$, then the natural number $-2z$ is mapped onto it; if $z = 0$, then 1 is mapped onto it; if $z > 0$, then $2z + 1$ is mapped onto it.

Consider the set of *rational numbers* ($\mathbb{Q}$). A rational number can be written as a ratio of integers where the denominator is not 0. We will focus on the positive rational numbers ($\mathbb{Q}^+$) which can be written as ratios of natural numbers. Since each natural number is a positive rational number, there is clearly a mapping from $\mathbb{Q}^+$ onto $\mathbb{N}$. Thus $card(\mathbb{Q}^+) \geq card(\mathbb{N})$.

There is also a mapping from $\mathbb{N}$ onto $\mathbb{Q}^+$. The diagram to the right shows a way of making a list of the



positive rational numbers that includes each positive rational number at least once.

Actually, each positive rational number is included more than once. For example, the rational number 2 appears in the list as $\frac{2}{1}$, $\frac{4}{2}$, $\frac{6}{3}$, $\frac{8}{4}$, etc.

We map 1 onto $\frac{1}{1}$, 2 onto $\frac{1}{2}$, 3 onto $\frac{2}{1}$, 4 onto $\frac{3}{1}$, 5 onto $\frac{2}{2}$, etc. Each positive rational number is the image of at least one natural number. So $card(\mathbb{N}) \geq card(\mathbb{Q}^+)$. Since it is also true that $card(\mathbb{Q}^+) \geq card(\mathbb{N})$, it follows that $card(\mathbb{Q}^+) = card(\mathbb{N}) = \aleph_0$. It can also be shown that $card(\mathbb{Q}) = \aleph_0$. So the number of rational numbers is the same as the number of natural numbers.

Numbers that can be plotted on the number line are called *real numbers* ($\mathbb{R}$). Let $\mathbb{R}^{(0,1)}$ symbolize the set of real numbers between 0 and 1. That set doesn't include 0 or 1, but does include all numbers between 0 and 1.

It can be shown[53] that there is a mapping from $\mathbb{R}^{(0,1)}$ onto $\mathbb{N}$. Thus $card(\mathbb{R}^{(0,1)}) \geq card(\mathbb{N})$. It can be shown[54] that it is impossible for there to be a mapping from $\mathbb{N}$ onto $\mathbb{R}^{(0,1)}$. Thus $card(\mathbb{R}^{(0,1)}) > card(\mathbb{N})$. The letter c is used to symbolize $card(\mathbb{R}^{(0,1)})$. So $c > \aleph_0$.

Consider the positive real numbers ($\mathbb{R}^+$). The numbers in $\mathbb{R}^{(0,1)}$ are also in $\mathbb{R}^+$. It follows from this that $card(\mathbb{R}^+) \geq card(\mathbb{R}^{(0,1)})$. It can also be shown[55] that $card(\mathbb{R}^{(0,1)}) \geq card(\mathbb{R}^+)$. So $card(\mathbb{R}^+) = card(\mathbb{R}^{(0,1)}) = c$.

It can also be shown that $card(\mathbb{R}) = card(\mathbb{R}^{(0,1)}) = c$. From the facts that $card(\mathbb{R}^{(0,1)}) = c$ and $card(\mathbb{N}) = \aleph_0$ and

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \cdots \\ \frac{1}{1} & \frac{1}{2} & \frac{2}{1} & \frac{3}{1} & \frac{2}{2} & \frac{1}{3} & \frac{1}{4} & \frac{2}{3} & \end{array}$$

$card(\mathbb{N}) \geq card(\mathbb{Q}^+)$

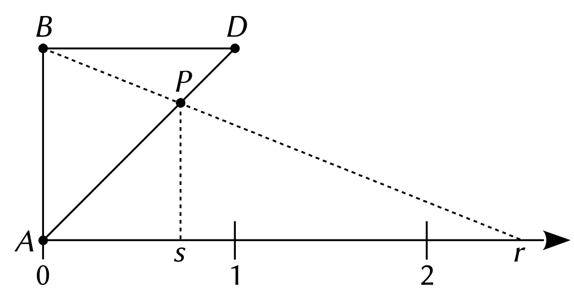
In this mapping, the rational number 1 is written as 1/1, 2/2, 3/3, etc. Thus many natural numbers are mapped onto the natural number 1. For example, 1, 5, and 13 are all mapped onto 1.

The real numbers between 0 and 1 form a *continuum*. The real numbers exist continuously between 0 and 1 without any breaks. A continuum's cardinality is called *c*. This *c* should not be confused with the *c* used for the speed of light (299,792,458 m/s).

$$\begin{array}{cccc} \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \\ \downarrow & \downarrow & \downarrow & \cdots \\ 1 & 2 & 3 & \end{array}$$

This mapping shows a way to make each natural number an image of at least one number between 0 and 1.

To the left is an outline of an *indirect proof*. Such a proof begins with a supposition that is the opposite of what is to be proved. Then logical reasoning results in a contradiction. This indicates that the supposition is false. So the opposite of the supposition must be true. In the Bible, the Apostle Paul used a type of indirect reasoning in I Corinthians 15:12-20.



$$s = \frac{r}{r+1} \qquad r = \frac{s}{1-s}$$

[53] If n is in $\mathbb{N}$, then $n \div (n+1)$ is in $\mathbb{R}^{(0,1)}$. Consider a mapping that maps $n \div (n+1)$ onto n and maps all other numbers in $\mathbb{R}^{(0,1)}$ onto 1. So each element of $\mathbb{N}$ is an image of at least one number in $\mathbb{R}^{(0,1)}$.

[54] Suppose there is a mapping from $\mathbb{N}$ onto $\mathbb{R}^{(0,1)}$. Symbolize the image of 1 by $0.a_{1,1}a_{1,2}a_{1,3}\ldots$ where $a_{1,1}$, $a_{1,2}$, and $a_{1,3}$ are the first 3 digits after the decimal point. Following a similar pattern, $0.a_{2,1}a_{2,2}a_{2,3}\ldots$ is the image of 2, etc.

Consider the number $0.b_1b_2b_3\ldots$ where the digits b_1, b_2, b_3, etc. are as follows: if $a_{1,1}$ is 3, then b_1 is 7, otherwise b_1 is 3; if $a_{2,2}$ is 3, then b_2 is 7, otherwise b_2 is 3; if $a_{3,3}$ is 3, then b_3 is 7, otherwise b_3 is 3; the pattern continues. So each b_n differs from $a_{n,n}$.

Observe that $0.b_1b_2b_3\ldots$ is not the image of 1 since b_1 doesn't equal $a_{1,1}$. Also, $0.b_1b_2b_3\ldots$ is not the image of 2 since b_2 doesn't equal $a_{2,2}$. In general, if n is a natural number, then $0.b_1b_2b_3\ldots$ is not the image of n since b_n doesn't equal $a_{n,n}$. So $0.b_1b_2b_3\ldots$ is in $\mathbb{R}^{(0,1)}$ and is not the image of any number in $\mathbb{N}$. So the mapping is not onto $\mathbb{R}^{(0,1)}$ which is contrary to the supposition. Therefore the supposition is false since it led to a contradiction. So there is not a mapping from $\mathbb{N}$ onto $\mathbb{R}^{(0,1)}$.

[55] The diagram to the right shows a geometrical way to view a mapping from $\mathbb{R}^{(0,1)}$ onto $\mathbb{R}^+$. Point D is directly above 1 on the number line. If r is a number in $\mathbb{R}^+$, then draw a line from r to point B. Call P the point where this line intersects diagonal line AD. Then draw a vertical line from P down to the number line. Call s the number at that point on the number line. The number s is in $\mathbb{R}^{(0,1)}$ and is mapped onto r in $\mathbb{R}^+$. Regardless of which number r is chosen from $\mathbb{R}^+$, there can always be found a unique number s in $\mathbb{R}^{(0,1)}$ that maps onto r. So there is a mapping from $\mathbb{R}^{(0,1)}$ onto $\mathbb{R}^+$. To the right are algebraic equations that show how to find s if r is known, or how to find r if s is known.

$c > \aleph_0$, it appears that there are more numbers between 0 and 1 than there are natural numbers!

If $\mathbb{T}_1 = \{1, 4, 7, 10, \ldots\}$, $\mathbb{T}_2 = \{2, 5, 8, 11, \ldots\}$, and $\mathbb{T}_3 = \{3, 6, 9, 12, \ldots\}$, then $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ have no numbers in common. Also, the union of $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ is $\mathbb{N}$. In other words, $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ together contain all of the natural numbers.

It can be shown that $card(\mathbb{T}_1) = \aleph_0$, $card(\mathbb{T}_2) = \aleph_0$, and $card(\mathbb{T}_3) = \aleph_0$. So each of these sets has the same cardinality as $\mathbb{N}$. Since $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$ have no elements in common, and since joining them forms $\mathbb{N}$, this suggests that $\aleph_0 + \aleph_0 + \aleph_0 = \aleph_0$. Similarly, it can be shown that $c + c + c = c$. The arithmetic of infinite numbers is quite different from the arithmetic of finite numbers. The symbol ∞ is often used to represent the concept of infinity. Whether ∞ refers to $\aleph_0$ or to c, we can write $\infty + \infty + \infty = \infty$.

Humility

Studying infinite numbers should point us to the greatness of our Creator. "How precious also are thy thoughts unto me, O God! how great is the sum of them! *If* I should count them, they are more in number than the sand...."[56] There are more grains of sand in the world than we can count. Thinking about sand grains points us to the concept of infinity. However, the number of grains of sand is finite, even though we are unable to count them. People might estimate the number, but not one of us can truly count them.

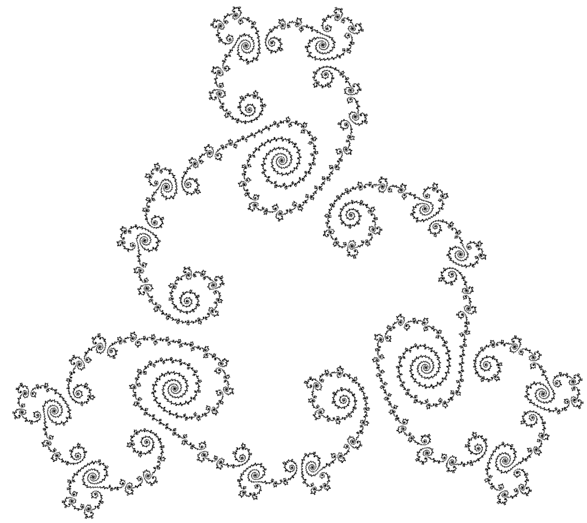
Mathematics can help us to better appreciate the greatness of our Creator. Studying His greatness should change us.[57] Mathematics should teach us humility. "O LORD our Lord, how excellent *is* thy name in all the earth! who hast set thy glory above the heavens. … When I consider thy heavens, the work of thy fingers, the moon and the stars, which thou hast ordained; What is man, that thou art mindful of him? and the son of man, that thou visitest him? For thou hast made him a little lower than the angels, and hast crowned him with glory and honour. Thou madest him to have dominion over the works of thy hands.... O LORD our Lord, how excellent *is* thy name in all the earth!"[58]

[56] Psalm 139:17-18a KJV.

[57] Consider how Job changed when God gave him a glimpse of "the panorama of His power in nature" (Larry L. Zimmerman, *Truth & the Transcendent* [Florence, KY: Answers in Genesis, 2000], 63). "But we all, with unveiled face, beholding as in a mirror the glory of the Lord, are being transformed...." (II Corinthians 3:18 NKJV).

[58] Psalm 8:1, 3-6a, 9 KJV.

For the invisible things of him from the creation of the world are clearly seen, being understood by the things that are made....
Romans 1:20a KJV



Above is the Julia set for $z^3 - 0.1 + 0.78i$. Its cardinality is c. Can you see how to divide this set into three pieces of equal size? The cardinality of each piece is c. This shows that $c + c + c = c$.

Before the mountains were brought forth, or ever thou hadst formed the earth and the world, even from everlasting to everlasting, thou *art* God.
Psalm 90:2 KJV

O come, let us worship and bow down: let us kneel before the LORD our maker.
Psalm 95:6 KJV

For *in* six days the LORD made heaven and earth, the sea, and all that in them *is*....
Exodus 20:11a KJV

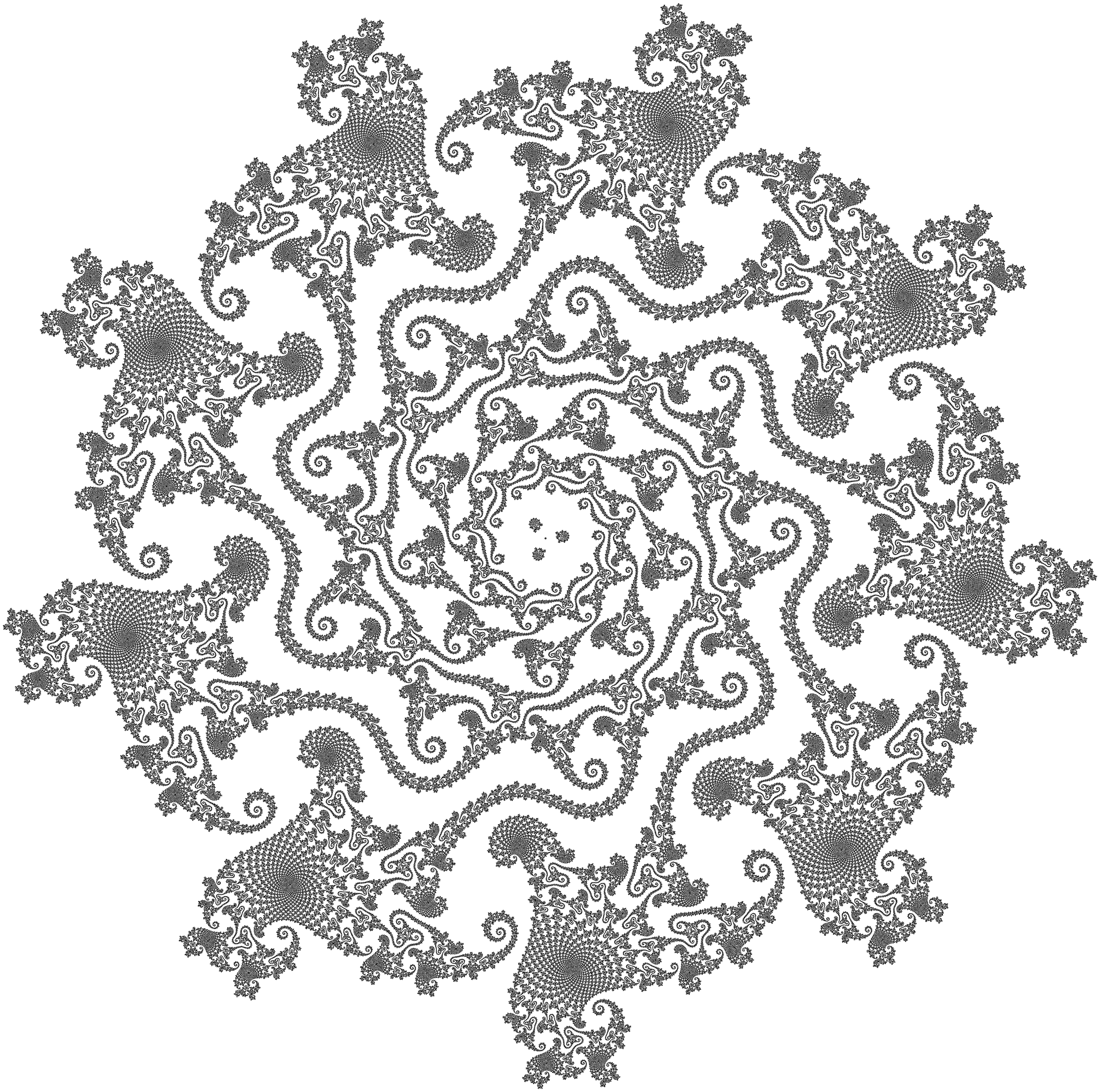
You are worthy, O Lord,
To receive glory and honor and power;
For You created all things,
And by Your will they exist and were created.
Revelation 4:11 NKJV

The writing of this document began in 2022. To learn how to contact the writer, use the mapping below on *qlsmist zg tnzro wlg xln.*

$$\begin{array}{c} a, b, c, \ldots \\ \downarrow\ \downarrow\ \downarrow \\ z, y, x, \ldots \end{array}$$

Thine, O Lord, *is* the greatness, and the power, and the glory, and the victory, and the majesty: for all *that is* in the heaven and in the earth *is thine*; thine *is* the kingdom, O Lord, and thou art exalted as head above all.

I Chronicles 29:11 KJV



This design is from the border of the Mandelbrot set for $z^3 + c$. The center of the design is near $0.3406608286508569105121947960888375 + 1.271325330373378670384814885616691i$. The height of the design is around 8×10^{-33}. Many thin filaments in this region are not shown.